The lattice quantum gravity, its continuum limit and the cosmological constant problem

S.N. Vergeles
Landau Institute for Theoretical Physics, Russian Academy of Sciences, Chernogolovka, Moskow region, 142432 Russia

Some variant of discrete quantum theory of gravity is constructed and its naive continual limit is considered. It is shown that this continual quantum theory of gravity leads to "light" universe comparatively with the universe in usual quantum theory. Thus in the theory the cosmological constant problem in inflating Universe has a natural solution.

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I. INTRODUCTION

Some time ago I have formulated some variant of discrete quantum gravity [1], and regularized continual quantum theory of gravity [2] – [4]. In the work [5] the arguments are given in favour of the discrete quantum gravity [1] has continual limit. The continual limit is described in [2] – [4]. In the present paper I show that in this theory the cosmological constant problem has natural solution.

Let's outline shortly the cosmological constant problem (the reader can find the review of the problem in [6]).

Consider Einstein equation with Λ-term ($\hbar = c = 1$):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T_{\mu\nu} + \Lambda \, g_{\mu\nu} \, . \tag{1.1}$$

Here $T_{\mu\nu}$ is energy-momentum tensor of the matter and Λ is some constant parameter having the dimension $[cm^{-2}]$. In the used unit system the Newtonian gravitational constant

$$G \sim l_P^2 \sim 2,5 \cdot 10^{-66} \, cm^2 \, , \tag{1.2}$$

and according to experimental data the mean energy density today is of the order

$$T_{\mu\nu} \sim \rho_1 \sim 10^8 cm^{-4} \longrightarrow 8\pi \, G \, T_{\mu\nu} \sim 5 \cdot 10^{-57} cm^{-2} \, , \tag{1.3}$$

and

$$\Lambda \sim 10^{-56} cm^{-2} \, . \tag{1.4}$$

Thus, if Einstein equation (1.1) is used for description of the today dynamics of Universe, the quantities in its right hand side are of the same order indicated in (1.3) and (1.4).

Now let us estimate the possible value of the right hand side of Eq. (1.1) in the framework of canonical quantum field theory. For simplicity consider energy-momentum tensor in quantum electrodynamics in flat spacetime:

$$T_{\mu\nu} = -\frac{1}{4\pi} \left(F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} \eta_{\mu\nu} F^2 \right) + \\ + \frac{i}{2} \left(\overline{\psi} \gamma_{(\mu} \nabla_{\nu)} \psi - \overline{\nabla_{(\nu} \psi} \gamma_{\mu)} \psi \right) \, . \tag{1.5}$$

Casimir effect, predicted in [7] and experimentally verified in [8], shows for reality of zero-point energies. Moreover, the attempts to drop out zero-point energies by appropriate normal ordering of creating and annihilating operators in energy-momentum tensor fail for many of reasons (the discussion of this problem see, for example, in [9]). Thus, at estimating vacuum expectation value of energy-momentum tensor (1.5), it should not be performed normal ordering of creating and annihilating operators in (1.5). Thus we obtain for vacuum expectation value of tensor (1.5) in free theory:

$$\langle T_{\mu\nu} \rangle_0 = \int \frac{\mathrm{d}^{(3)} \, k}{(2\pi)^3} \left(\left. \frac{k_\mu k_\nu}{k^0} \right|_{k^0 = |\,\mathbf{k}\,|} - \right. \\ \left. - \left. \frac{2 k_\mu k_\nu}{k^0} \right|_{k^0 = \sqrt{m^2 + \mathbf{k}^2}} \right) \, . \tag{1.6}$$

Here m is the electron mass. The first item in (1.6) gives the positive contribution but the second item gives the negative contribution since these items give the boson and fermion contributions to vacuum energy, respectively. If integration in (1.6) is restricted by Planck scale, $k_{max} \sim l_P^{-1}$, then from (1.6) and (1.2) it follows:

$$8\pi G \langle T_{\mu\nu} \rangle_0 \sim l_P^{-2} \sim 10^{66} \, cm^{-2} \, . \tag{1.7}$$

It is clear that the interaction of fields doesn't changes qualitatively the estimation (1.7). From (1.7) and (1.3) we see that the contribution to the righthand side of Eq. (1.1) estimated in the framework of canonical quantum field theory is larger about 10^{120} times in comparison with the experimental estimations.

It is known that in globally supersimmetric field theories the vacuum energy is equal to zero [10]. Indeed, in flat spacetime the anticommutation relations

$$\{ Q_\alpha, \, Q_\beta^\dagger \} = (\sigma_\mu)_{\alpha\beta} \mathcal{P}^\mu \, . \tag{1.8}$$

take place. Here Q_α are supersimmetry generators, α β are spinor indexes, σ_1, σ_2 and σ_3 are the Pauli matrices, $\sigma_0 = 1$, and $\mathcal{P}^\mu$ is the energy-momentum 4-vector operator. If supersimmetry is unbroken, then the vacuum state $|0\rangle$ satisfies

$$Q_\alpha |0\rangle = Q_\alpha^\dagger |0\rangle = 0 \, , \tag{1.9}$$

and (1.8) and (1.9) imply

$$\langle \mathcal{P}^{\mu} \rangle_0 = 0 \,. \tag{1.10}$$

The equality (1.10) means that the total sum of zero-point energies in unbroken globally supersimmetric field theories is rigorously equal to zero.

However, even if supersimmetry takes place on fundamental level, it is broken on experimentally tested scales. If one assumes that supersimmetry is unbroken on the scales greater than $k_{SS} \sim 10^{17}\,cm^{-1}\,(\sim 10^3\,GeV)$, then even in this case the contribution to the right hand side of Eq. (1.1) from zero-point energies of all normal modes with energies less then k_{SS} will exceed experimentally known value about 10^{58} times.

It follows from the said above that any calculation in the framework of canonical quantum field theory leads to unacceptable large vacuum expectation value of energy momentum tensor. The considered catastrophe isn't solved at present in superstring theory.

It should be noted here that the problem of cosmological constant is solved in original theory of G. Volovik [11]. In this theory the gravitons and other excitations are the quasiparticles in a more fundamental quantum system — quantum fluid of the type ${}^3He$ in superfluid phase. Another approach to the problem of cosmological constant in the frame of M-theory is developed in works [12].

This paper is organized as follows. In sect. 2 I define discrete quantum gravity which has been introduced in [1]. It is shown qualitatively that this quantum theory display the tendency to degenerate into macroscopic continual theory and the continual limit in this discrete theory is possible. Using high-temperature expansion (which is possible at small times near the singularity) the important for the following consideration conclusion is made that the two-point gauge invariant correlators of any fields are local, i.e. they decrees exponentially in x-space. From here and the dynamics of discrete theory the interesting conclusion about noncompact packing of field modes in momentum space in the continual theory is made. In sects. 3 and 4 formulation of the corresponding continual quantum theory of gravity and its connection with discrete theory is given [1] – [5]. Naturally, the structure of the continual theory is determined by more fundamental discrete theory. In sect. 5 it is shown that in the framework of the suggested quantum theory of gravity the cosmological constant problem can be solved.

II. DISCRETE QUANTUM GRAVITY

A. Definition of Action

Let $\mathfrak{K}$ be a 4-dimensional simplicial complex admitting gemetrical realization. The definition and required properties of simplicial complexes can be found in [1]. A detailed theory of simplicial complexes is given, for example, in [13] – [14]. Below instead of "simplicial complex" we say simply "complex", and the concepts in the following pairs are treated as synonyms: 0-simplex and vertex; 1-simplex and edge; 2-simplex and triangle; 3-simplex and tetrahedron. The finite complexes with a 4-disk topology are interesting here. Such complexes have a boundary $\partial\,\mathfrak{K}$ which is 3-dimensional complex with topology of 3-sphere S^3. Denote by $\alpha_q,\ q = 0, 1, 2, 3, 4$ the number of q-simplexes of the complex $\mathfrak{K}$. The indexes $i, j, k, l, \ldots$ run through the complex vertices: $a_i,\ a_j$ and so on. Two vertices are called adjacent if these two vertices are the boundary vertices of the same edge.

For convenience I give here the definition of orientation of simplexes and complexes.

A simplex

$$s^r = \varepsilon(a_0,\, a_1,\, \ldots\,,\, a_r) \equiv \varepsilon\, a_0\, a_1\, \ldots\, a_r \tag{2.1}$$

has an orientation, or is oriented, if every order of its vertices is assigned a sign "+" or "-", so that orders differing by an odd permutation correspond to opposite signs. Thus if $\varepsilon = 1$ the orientation of simplex (2.1) is given by the orders $(a_0,\, a_1,\, \ldots\,,\, a_r)$ ore $-(a_1,\, a_0,\, \ldots\,,\, a_r)$. Let $(a_0,\, \ldots\,,\, a_{i-1},\, a_{i+1},\, \ldots\,,\, a_r)$ be the face of a simplex s^r obtained by eliminating one vertex a_i from the sequence $a_0,\, a_1,\, \ldots\,,\, a_r$. By definition, the orientation of this face, given by

$$B_i^{r-1} = (-1)^i\, \varepsilon(a_0,\, \ldots\,,\, a_{i-1},\, a_{i+1},\, \ldots\,,\, a_r)\,, \tag{2.2}$$

is called an iduced orientation of the simplex s^r.

Denote by D the maximum value of number r in (2.1) for all simplexes of complex. In considered case $D = 4$. Thus D is the dimension of complex. Two oriented D-dimensional simplices s_1^D and s_2^D of a D-dimensional simplicial complex are called concordantly oriented if either the simplices s_1^D and s_2^D have no common $(D-1)$-dimensional faces or the orientation of their common $(D-1)$-dimensional face B^{D-1} induced by the orientation of the simplex s_1^D is opposite to the orientation of the same face B^{D-1} induced by the orientation of the simplex s_2^D. A D-dimensional simplicial complex $\mathfrak{K}$ is called orientable if there exists such an orientation for all its D-dimensional simplices that any pair of its D-dimensional simplices is concordantly oriented. The concordant orientation of D-dimensional simplices defines the orientation of the complex, and namely this orientation of D-simplices is regarded as positive.

Evidently, interesting for us complex $\mathfrak{K}$ is orientable.

Below index A enumerates 4-simplices. Introduce the following notation for oriented 1-simplices in the case when the vertexes a_i and a_j belong to the 4-simlex with index A:

$$X_{ij}^A = a_i a_j = -X_{ji}^A \,. \tag{2.3}$$

Let

$$s_A^4 = a_{i_0} a_{i_1} a_{i_2} a_{i_3} a_{i_4} \tag{2.4}$$

be an positively oriented 4-simlex with index A. An oriented frame of a simplex (2.4) at a vertex a_{i_0} is the ordered set of four oriented 1-simplices (2.3) such that an even permutation of these 1-simplices does not change the orientation while an odd permutation changes the orientation of the frame to the opposite. By definition, the frame

$$\mathcal{R}^{A\,i_0} = \left(X^A_{i_0 i_1},\, X^A_{i_0 i_2},\, X^A_{i_0 i_3},\, X^A_{i_0 i_4}\right) \tag{2.5}$$

is oriented positively.

Let $\gamma^a,\ a,b,c,\ldots = 1,2,3,4$ be 4×4 Dirac matrices with Euclidean signature. Thus all Dirac matrices as well matrix

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4\,, \qquad \mathrm{tr}\ \gamma^5\gamma^a\gamma^b\gamma^c\gamma^d = 4\,\varepsilon^{abcd} \tag{2.6}$$

are Hermitian. To each vertex a_i, we assign the Dirac spinors ψ_i and $\overline{\psi}_i$ each of whose components assumes values in a complex Grassman algebra. In the case of Euclidean signature, the spinors ψ_i and $\overline{\psi}_i$ are independent variables and are interchanged under the Hermitian conjugation. The Dirac matrixes act from the left to the spinors ψ_i and from the right to the spinors $\overline{\psi}_i$.

Let us assign to each oriented edge $a_i a_j$ an element of the group $Spin(4)$:

$$\Omega_{ij} = \Omega^{-1}_{ji} = \exp\left(\frac{1}{2}\omega^{ab}_{ij}\sigma^{ab}\right)\,, \quad \sigma^{ab} = \frac{1}{4}[\gamma^a,\,\gamma^b]\,. \tag{2.7}$$

Holonomy element Ω_{ij} of the gravitational field executes a parallel transformation of spinor ψ_j from vertex a_j of edge $a_i a_j$ to neighboring vertex a_i. We denote by V a linear space with basis γ^a. Let each oriented edge $a_i a_j$ be put in correspondence with element $\hat{e}_{ij} \equiv e^a_{ij}\gamma^a \in V$, such that

$$\hat{e}_{ij} = -\Omega_{ij}\hat{e}_{ji}\Omega^{-1}_{ij}\,. \tag{2.8}$$

The notations $\overline{\psi}_{Ai},\ \psi_{Ai},\ \hat{e}_{Aij},\ \Omega_{Aij}$ and so on indicate that edge $X^A_{ij} = a_i a_j$ belongs to 4-simplex with index A. Here, the sign " $-$ " in (2.8) is due to the fact that $e_{A\,ij}$ and $e_{A\,ji}$ are the values of the 1-form on the edges $X^A_{ij} = a_i a_j$ and $X^A_{ji} = a_j a_i = -a_i a_j = -X^A_{ij}$ (which are oriented mutually oppositely), respectively. The "facing" from the elements of a holonomy group on the right-hand side of Eq. (2.8) are necessary since the element $e_{A\,ji}$ must be paralled-translated from the vertex $a_{A\,j}$ to the vertex $a_{A\,i}$ to compare this element with the element $e_{A\,ij}$. The quantities assigned to each oriented edge $a_i a_j$ and satisfying to Eq. (2.8) are called 1-forms.

By assumption, complex $\mathfrak{K}$ has a disk topology. For such a complex, the concept of orientation can be introduced. We define the orientation of the complex by defining the orientation of each 4-simplex. In this case, if two 4-simplices have a common tetrahedron, the two orientations of the tetrahedron, which are defined by the orientations of these two 4-simplices, are opposite. In our case, the complex obviously has only two orientations.

Let $a_{Ai},\ a_{Aj},\ a_{Ak},\ a_{Al}$, and a_{Am} be all five vertices of a 4-simplex with index A and $\varepsilon_{Aijklm} = \pm 1$ depending on whether the order of vertices $a_{Ai}\,a_{Aj}\,a_{Ak}\,a_{Al}\,a_{Am}$ defines the positive or negative orientation of this 4-simplex. In addition, $\varepsilon_{ijklm} = 0$ if at least two indices coincide. We can now write the Euclidean action in the model in question:

$$\begin{aligned} I = \frac{1}{5\times 24}\sum_A \sum_{i,j,k,l,m} \varepsilon_{Aijklm}\ \mathrm{tr}\ \gamma^5 \times \\ \times\Bigg\{-\frac{1}{2\,l_P^2}\Omega_{Ami}\Omega_{Aij}\Omega_{Ajm}\hat{e}_{Amk}\hat{e}_{Aml} + \\ +\frac{1}{24}\hat{\Theta}_{Ami}\hat{e}_{Amj}\hat{e}_{Amk}\hat{e}_{Aml}\Bigg\}\,, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \hat{\Theta}_{Aij} = \frac{i}{2}\gamma^a\left(\overline{\psi}_{Ai}\gamma^a\Omega_{Aij}\psi_{Aj} - \overline{\psi}_{Aj}\Omega_{Aji}\gamma^a\psi_{Ai}\right) \equiv \\ \equiv \Theta^a_{Aij}\gamma^a \in V\,. \end{aligned} \tag{2.10}$$

The quantity $\hat{\Theta}_{Aij}$ represents an Hermithian operator. One can easily verify that 1-form (2.10), just as the 1-form $\hat{e}_{ij}$, satisfies relation (2.8). This fact is established by the repeated application of the formula

$$S^{-1}\,\gamma^a\,S = S^a_b\,\gamma^b\,, \tag{2.11}$$

where

$$\begin{aligned} S \equiv \exp\frac{1}{2}\,\varepsilon_{ab}\,\sigma^{ab}\,, \qquad \varepsilon_{ab} = -\varepsilon_{ba} = \varepsilon^a_b\,, \\ S^a_b \equiv \left(\exp\varepsilon\right)^a_b = \delta^a_b + \varepsilon^a_b + \frac{1}{2}\,\varepsilon^a_c\varepsilon^c_b + \ \ldots\,. \end{aligned} \tag{2.12}$$

It is easy to see that the action (2.9) is real.

The volume of a 4-complex is given by

$$\begin{aligned} V_A = \frac{1}{4!}\times\frac{1}{5!}\times \\ \times\sum_A\sum_{i,j,k,l,m} \varepsilon_{A\,ijklm}\,\varepsilon^{abcd}\,e^a_{A\,mi}e^b_{A\,mj}e^c_{A\,mk}e^d_{A\,ml}\,. \end{aligned} \tag{2.13}$$

Here, factor $1/4!$ is required since the volume of a four-dimensional parallelepiped with generatrices $e^a_{A\,mi},\ e^b_{A\,mj},\ e^c_{A\,mk}$, and $e^d_{A\,ml}$ is $4!$ times larger than the volume of a 4-simplex with the same generatrices, while factor $1/5!$ is due to the fact that all five vertices of each simplex are taken into account independently.

The dynamic variables are quantities Ω_{ij} and $\hat{e}_{ij}$, which describe the gravitational degrees of freedom, and fields $\overline{\psi}_i$ and ψ_i, which are material fermion fields (other material fields are not considered here).

In the space of fields, there acts a gauge group according to the following rule. To each vertex $a_{A\,i}$, let us assign an element of the group $S_{A\,i} \in \mathrm{Spin}(4)$. According to the principle of gauge invariance, the fields $\Omega,\ e,\ \psi$, and the

transformed fields

$$\tilde{\Omega}_{A\,ij} = S_{A\,i}\,\Omega_{A\,ij}\,S^{-1}_{A\,j}\,,$$
$$\tilde{e}_{A\,ij} = S_{A\,i}\,e_{A\,ij}\,S^{-1}_{A\,i}\,,$$
$$\tilde{\psi}_{A\,i} = S_{A\,i}\,\psi_{A\,i}\,, \qquad \tilde{\overline{\psi}}_{A\,i} = \overline{\psi}_{A\,i}\,S^{-1}_{A\,i} \tag{2.14}$$

are physucally equivalent. This means that the action (2.9) is invariant under the transformations (2.14). Under the gauge transformatios (2.14), the 1-form Θ is transformed in the same way as the form e:

$$\tilde{\hat{\Theta}}_{A\,ij} = S_{A\,i}\,\hat{\Theta}_{A\,ij}\,S^{-1}_{A\,i}\,. \tag{2.15}$$

The last formula is verified with the help of Eqs. (2.11), (2.12) and (2.14). Gauge invariance the action (2.9) and the volume (2.13) is established by using Eqs. (2.14) and (2.15).

It is natural to interpret the quantity

$$l^2_{ij} \equiv \frac{1}{4}\,\mathrm{tr}\,(\hat{e}_{ij})^2 = \sum_{a=1}^{4}(e^a_{ij})^2 \tag{2.16}$$

as the square of the length of the edge $a_i a_j$. Thus, the geometric properties of a simplicial complex prove to be completely defined.

Now, let us show that, in the limit of slowly varying fields, the action (2.9) reduces to the continuum action of gravity, minimally connected with with a Dirac field, in a four-dimensional Euclidean space.

Consider a certain subset of vertices from the simplicial complex and assign the coordinates (real numbers)

$$x^\mu_{A\,i} \equiv x^\mu(a_{A\,i})\,, \qquad \mu = 1,\,2,\,3,\,4 \tag{2.17}$$

to each vertex $a_{A\,i}$ from this subset. We stress that these coordinates are defined only by their vertices rather than by the higher dimension simplices to whom these vertices belong; moreover, the correspondence between the vertices from the subset considered and the coordinates (2.17) is one-to-one.

Suppose that

$$|\,x^\mu_{A\,i} - x^\mu_{A\,j}\,| \ll 1\,. \tag{2.18}$$

Estimates (2.18) can easily be satisfied if the complex contains a vary large number of simplices and its geometric realization is an almost smooth four-dimensional surface [19]. Suppose also that the four 4-vectors

$$\mathrm{d}\,x^\mu_{A\,ji} \equiv x^\mu_{A\,i} - x^\mu_{A\,j}\,, \qquad i \neq j\,, \quad i = 1,\,2,\,3,\,4 \tag{2.19}$$

are linearly independent and

$$\begin{vmatrix} \mathrm{d}\,x^1_{A\,m1} & \mathrm{d}\,x^2_{A\,m1} & \dots & \mathrm{d}\,x^4_{A\,m1} \\ \dots & \dots & \dots & \dots \\ \mathrm{d}\,x^1_{A\,m4} & \mathrm{d}\,x^2_{A\,m4} & \dots & \mathrm{d}\,x^4_{A\,m4} \end{vmatrix} > 0\,, \tag{2.20}$$

provided that the frame $\left(X^A_{m\,1},\,\dots\,,\,X^A_{m\,4}\right)$ is positively oriented. Inequality (2.20) implies that positively oriented local coordinates are introduced on the almost flat surface considered. Here, the differentials of coordinates (2.19) correspond to one-dimensional simplices $a_{A\,j}a_{A\,i}$, so that, if the vertex $a_{A\,j}$ has coordinates $x^\mu_{A\,j}$, then the vertex $a_{A\,i}$ has the coordinates $x^\mu_{A\,j} + \mathrm{d}\,x^\mu_{A\,ji}$.

In the continuum limit, the holonomy group elements (2.7) are close to the identity element, so that the quantities ω^{ab}_{ij} tend to zero being of the order of $O(\mathrm{d}\,x^\mu)$. Thus one can consider the following system of equation for $\omega_{A\,m\mu}$:

$$\omega_{A\,m\mu}\;\mathrm{d}\,x^\mu_{A\,mi} = \omega_{A\,mi}\,, \quad i = 1,\,2,\,3,\,4\,. \tag{2.21}$$

In this system of linear equation, the indices A and m are fixed, the summation is carried out over the index μ, and index runs over all its values. Since the determinant (2.20) is positive, the quantities $\omega_{A\,m\mu}$ are defined uniquely. Suppose that a one-dimensional simplex $X^A_{m\,i}$ belong to four-dimensional simplices with indices $A_1,\,A_2,\,\dots\,,\,A_r$. Introduce the quantity

$$\omega_\mu\left[\frac{1}{2}\,(x_{A\,m} + x_{A\,i})\right] \equiv \frac{1}{r}\left\{\omega_{A_1\,m\mu} + \,\dots\, + \omega_{A_r\,m\mu}\right\}, \tag{2.22}$$

which is assumed to be related to the midpoint of the segment $[x^\mu_{A\,m},\,x^\mu_{A\,i}]$. Recall that the coordinates $x^\mu_{A\,i}$ just as the differentials (2.19), depend only on vertices but not on the higher dimensional simplices to which these vertices belong. According to the definition, we have the following chain of equalities:

$$\omega_{A_1\,mi} = \omega_{A_2\,mi} = \,\dots\, = \omega_{A_r\,mi}\,. \tag{2.23}$$

It follows from (2.19) and (2.21)–(2.23) that

$$\omega_\mu\left(x_{A\,m} + \frac{1}{2}\,\mathrm{d}\,x_{A\,mi}\right)\mathrm{d}\,x^\mu_{A\,mi} = \omega_{A\,mi}\,. \tag{2.24}$$

The value of the field ω_μ in (2.24) on each one-dimensional simplex is uniquely defined by this simplex.

Next, we assume that the fields ω_μ smoothly depend on the points belonging to the geometric realization of each four-dimensional simplex. In this case, the following formula is valid up to $O\big((\mathrm{d}\,x)^2\big)$ inclusive:

$$\Omega_{A\,mi}\,\Omega_{A\,ij}\,\Omega_{A\,jm} = \exp\left[\frac{1}{2}\,\mathfrak{R}_{A\,m\mu\nu}\;\mathrm{d}\,x^\mu_{A\,mi}\;\mathrm{d}\,x^\nu_{A\,mj}\right], \tag{2.25}$$

where

$$\mathfrak{R}_{A\,m\mu\nu} = \partial_\mu\omega_{A\,m\nu} - \partial_\nu\omega_{A\,m\mu} + [\omega_{A\,m\mu},\,\omega_{A\,m\nu}\,]\,. \tag{2.26}$$

On the right-hand side of (2.25), as well as in equality (2.26), all fields are taken at the vertex $a_{A\,m}$ of a four-dimensional simplex A as is indicated by the subscript

$A\,m$. When deriving formula (2.25), we used the Hausdorff formula.

In exact analogy with (2.21), let us write out the following relations for a tetrad field without explanations:

$$e_{A\,m\mu}\;\mathrm{d}\,x^{\mu}_{A\,mi} = e_{A\,mi}\,. \tag{2.27}$$

Using (2.7) and (2.21), we can rewrite the 1-form (2.10) as

$$\Theta_{A\,ij} = \gamma^a\,\frac{i}{2}\,\left[\,\overline{\psi}_{A\,i}\,\gamma^a\,\mathcal{D}_\mu\,\psi_{A\,i} - \overline{\mathcal{D}_\mu\,\psi}_{A\,i}\,\gamma^a\,\psi_{A\,i}\,\right]\,\mathrm{d}\,x^{\mu}_{A\,ij}\,, \tag{2.28}$$

to within $O(\mathrm{d}\,x)$; here,

$$\mathcal{D}_\mu\,\psi_{A\,i} = \partial_\mu\,\psi_{A\,i} + \omega_{A\,i\mu}\,\psi_{A\,i}\,. \tag{2.29}$$

Before rewriting the action (2.9) in the continuum limit, we give the following obvious formula:

$$\sum_{\sigma(Am)} \varepsilon_{\sigma(Am)}\;\mathrm{d}\,x^{\mu}_{A\,mi}\;\mathrm{d}\,x^{\nu}_{A\,mj}\;\mathrm{d}\,x^{\lambda}_{A\,mk}\;\mathrm{d}\,x^{\rho}_{A\,ml} = \\ = 24\,\varepsilon^{\mu\nu\lambda\rho}\,v_{S\,A}\,. \tag{2.30}$$

Here, $\varepsilon^{\mu\nu\lambda\rho}$ is a completely antisymmetric symbol, which is equal to unity when $(\mu\,\nu\,\lambda\,\rho) = (1\,2\,3\,4)$ (compare with (2.20)), and $v_{S\,A}$ is the volume of the geometric realization of simplex A in a four-dimensional Euclidean space when the Euclidean coordinates of the geometric realization of the simplex are equal to the corresponding coordinates of its vertices (2.17). The factor 24 in (2.30) is necessary since the volume $v_{S\,A}$ of the four-dimensional simplex on the right-hand side is less than the volume of a four-dimensional parallelepiped constructed on the vectors $\mathrm{d}\,x^{\mu}_{A\,mi}$, … , $\mathrm{d}\,x^{\mu}_{A\,ml}$ by a factor of 24.

Applying formulas (2.25)–(2.30) and changing the summation to integration, we obtain the following expression for the action (2.9) in the continuum limit:

$$I = \int\,\mathrm{tr}\,\gamma^5 \times \\ \times \left[-\frac{1}{4\,l_P^2}\left(\mathfrak{R} + \frac{1}{3}\,\Lambda\,e\wedge e\right) + \frac{1}{24}\,\Theta\wedge e\right]\wedge e\wedge e\,. \tag{2.31}$$

Here, the curvature 2-form (see (2.26)) and the 1-forms (see (2.27), (2.28)) are defined by

$$\mathfrak{R} \equiv \frac{1}{2}\,\sigma^{ab}\,R^{ab}_{\mu\nu}\;\mathrm{d}\,x^\mu\wedge\mathrm{d}\,x^\nu\,, \\ e = \gamma^a\,e^a_\mu\;\mathrm{d}\,x^\mu\,, \\ \Theta = \gamma^a\,\frac{i}{2}\left[\,\overline{\psi}\gamma^a\,\mathcal{D}_\mu\,\psi - \overline{\mathcal{D}_\mu\psi}\,\gamma^a\,\psi\,\right]\,\mathrm{d}\,x^\mu\,. \tag{2.32}$$

Thus, in the continuum limit, the action (2.9) proves to be equal to the action of gravity with a Λ-term and a metric with Euclidean signature that is minimally connected with a Dirac field.

B. Quantization of Discrete Gravity

Let us determine the partition function Z for a discrete Euclidean gravity, which becomes the transfer-matrix in discrete quantum gravity after passing to the Lorentzian signature. Let us enumerate the zero-dimensional (vertices) and one-dimensional (edges) simplices by indices $\mathcal{V}$ and $\mathcal{E}$, respectively, and denote by $\psi_{\mathcal{V}}$, $\Omega_{\mathcal{E}}$, rtc. the corresponding variables. By definition,

$$Z = \mathrm{const}\cdot\left(\prod_{\mathcal{E}}\int\,\mathrm{d}\,\Omega_{\mathcal{E}}\int\,\mathrm{d}\,e_{\mathcal{E}}\right)\times \\ \times\left(\prod_{\mathcal{V}}\mathrm{d}\,\overline{\psi}_{\mathcal{V}}\;\mathrm{d}\,\psi_{\mathcal{V}}\right)\,\exp\left(-I\right)\,. \tag{2.33}$$

Here, const is a certain normalizing factor, $\mathrm{d}\,\Omega_{\mathcal{E}}$ is the Haar measure on the group Spin(4),

$$\mathrm{d}\,e_{\mathcal{E}} \equiv \prod_a\,\mathrm{d}\,\omega^a_{\mathcal{E}}\,, \qquad e_{\mathcal{E}} = \omega^a_{\mathcal{E}}\,\gamma^a\,, \tag{2.34}$$

and

$$\mathrm{d}\,\overline{\psi}_{\mathcal{V}}\;\mathrm{d}\,\psi_{\mathcal{V}} \equiv \prod_\nu\,\mathrm{d}\,\overline{\psi}_{\mathcal{V}\nu}\;\mathrm{d}\,\psi_{\mathcal{V}\nu}\,. \tag{2.35}$$

The index ν in (2.35) enumerates individual components of the spinors $\psi_{\mathcal{V}}$ and $\overline{\psi}_{\mathcal{V}}$, such that we have a product of the differentials of all independent generators of the Grassman algebra of Dirac spinors in (2.35). The action I in (2.33) is defined by formula (2.9).

Note that the measure (2.34) is determined correctly in view of invariance of the Haar measure and the relations (2.7) and (2.8). Therefore, one can really assume that the measure (2.34) is related to the set of edges.

Obviously, all the measures used in the functional integral (2.33) are invariant under the gauge transformations (2.14). Since the action I (2.9) in (2.33) is also gauge invariant, the partition function (2.33) is invariant under the action of the gauge group (2.14).

Consider the partition function (2.33) with a zero Λ-term in the absence of fermions. In this case, the integral over the 1-form $e_{\mathcal{E}}$ becomes Gaussian:

$$Y\{\,\Omega\,\} = \int\,D\,z\cdot\exp\left(\frac{1}{2}\,z_m\,\mathcal{M}_{m\,n}\,z_n\right)\,. \tag{2.36}$$

Here, $\{\,z_m\,\}$, $m = 1,\,\dots\,,\,Q$ denotes a set of real variables $\{\omega^a_{\mathcal{E}}\}$ and $\mathcal{M}_{mn}$ is a real symmetrical matrix depending on the elements of the holonomy group $\Omega_{\mathcal{E}}$. Thus,

$$\frac{1}{2}\,z_m\,\mathcal{M}_{mn}\,z_n \equiv \frac{1}{l_P^2}\,\frac{1}{5}\cdot\frac{1}{24}\sum_{A,\,m}\,\sum_{\sigma(A\,m)}\varepsilon_{\sigma(Am)}\times \\ \times\,\mathrm{tr}\left(\gamma^5\,\Omega_{A\,mi}\,\Omega_{A\,ij}\,\Omega_{A\,jm}\,e_{A\,mk}\,e_{A\,ml}\right)\,. \tag{2.37}$$

Denote by $\{\,\lambda_q\,\}$, where $q = 1,\,\dots\,,\,Q$, a set of eigenvalues of the matrix $\mathcal{M}_{mn}$. Let $\varepsilon_q = \mathrm{sign}\,\lambda_q$. Since, in

general, there are both negative and positive eigenvalues among $\{\lambda_q\}$, the integral (2.36) should be redefined. This is done by passing to Lorentzian signature. Under this procedure, the eigenvalues are transformed by the rule

$$\lambda_q \to e^{i\varphi}\,\lambda_q\,,$$

where $\varphi = 0$ in the Euclidean space and $\varphi = \pi/2$ in the case of the Minkowski signature. Thus, the Euclidean Gaussian integral

$$\mathcal{I}_E = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\,z \cdot \exp\left(\frac{1}{2}\,\lambda\, z^2\right) \tag{2.38}$$

reduces to the Fresnel integral in the Minkowski signature:

$$\mathcal{I}_M = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\,z \cdot \exp\left(\frac{i}{2}\,\lambda\, z^2\right) = \sqrt{\frac{i}{\lambda}} = \\ = (i)^{\frac{\varepsilon}{2}}\,\frac{1}{\sqrt{|\,\lambda\,|}}\,, \tag{2.39}$$

where $\varepsilon = \operatorname{sign}\lambda$. Let us perform the analytic continuation

$$\lambda \to e^{-i\varphi}\,\lambda$$

on the right-hand side of Eq. (2.39) and set $\varphi = \pi/2$. Thus, we recover the Euclidean signature of a metric and obtain the following value for integral (2.38):

$$\mathcal{I}_E = (i)^{\frac{\varepsilon+1}{2}}\,\frac{1}{\sqrt{|\,\lambda\,|}}\,. \tag{2.40}$$

Now, using Eq. (2.40), we redefine the integral (2.36) of interest:

$$Y\{\Omega\} = \mathrm{const}\,\prod_q (i)^{\frac{\varepsilon_q+1}{2}}\,|\,\lambda_q\,|^{-1/2}\,. \tag{2.41}$$

If there are fermion fields in the theory, one should first calculate a functional integral over fermions. The subsequent integration over the 1-form e remains Gaussian and yields a contribution of the form (2.41) to the partition function. The remaining integral over the elements of the holonomy group Ω may prove to be divergent despite the compactness of this group. Indeed, certain eigenvalues λ_q may vanish under certain configurations of the field Ω. Since the expression under the integral sign depends on the negative powers of λ_q, the integral over the field Ω may prove to be divergent. From the physical point of view, these divergences are of great interest. Note that the tendency of eigenvalues λ_q to zero implies that the integral over the 1-form e^a is saturated when the absolute values of this field e^a (or its certain components) tend to infinity. This means that the size of universe tends to infinity (see (2.16)). On the other hand, as will be shown below, the fact that the field components e^a have large values implies that the dynamics of the system becomes quasiclassical. Therefore, from the physical viewpoint, these divergences imply birth of quasiclassical macroscopic space-time.

Concerning the problem under discussion, we note that the presence of Dirac fields in integral (2.33) only strengthens the divergence under the integration over the field e^a. Indeed, after the integration over the fermion field, the integral over the field e^a is rewritten as (cf. (2.38) and (2.39))

$$\mathcal{I} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\,z\, P_n(z) \cdot \exp\left(\frac{i}{2}\,\lambda z^2\right)\,, \tag{2.42}$$

where $P_n(z)$ is a polynomial in z of degree n. For small λ, integral (2.42) is proportional to $|\,\lambda\,|^{-(n+1)/2}$.

A similar physical interpretation of divergences under the integration over the field e^a in the continuum quantum B-F-theory in a three-dimensional space-time was given by Witten in [15].

Let us notice another possible type of divergences in a discrete quantum gravity. If the partition function (2.33) was defined for a metric with Lorentzian signature, then the elements of the holonomy group would be the noncompact group Spin(3, 1). The gauge group (2.14) would also be noncompact, being a direct product of the $\mathcal{V}$ copies of the group Spin(3, 1). Since both the measure and action in the transfer-matrix are gauge invariant, the functional integral in the transfer-matrix would not be defined at all before the fixation (at least partial) of the gauge. However, the fixation of the gauge in the fundamental transfer-matrix seems to be a so artificial procedure that the theory itself looses its beauty and sense. In our opinion, this means that the fundamental partition function for a discrete theory of gravity can be constructed only on the basis of a metric with Euclidean signature.

In their well-known paper [16], Hartle and Hawking made a hypothesis that the wave function of the universe must be calculated with the use of the functional integral on the basis of a metric with Euclidean signature. But in the case of the gravity theory the Euclidean action is not positively defined. In our opinion, the arguments for a metric with Euclidean signature provided by the discrete theory of gravity are much more reliable than the arguments given in [16].

C. High Temperature Expansion

From the beginning let us consider the integral (2.33) in the region of integration variables where

$$|e^a_{ij}| > l_0 \gg l_P\,. \tag{2.43}$$

In this region each item in the sum (2.9) generally is also large since the items in the sum (2.9) are polynomials in the variables e^a_{ij} of powers not less than two. Therefore the whole integral in (2.33) can be estimated quasiclassically ore by the stationary phase method. In this region one must use the long wave limit action (2.31), and to

perform the stationary phase calculations the integration paths in (2.33) must be deformed so that Lorentzian signature is realized. Thus the time arises. We see that in considered model the time arises dynamically in continual limit. The study of continual limit of the theory is performed in the subsequent sections.

Now let us consider the integral (2.33) in the region of integration variables where

$$|e^a_{ij}| < l_1 \ll l_P \, . \tag{2.44}$$

In this region each item in the sum (2.9) is small, so that the subintegral quantity in (2.33) (in the case of pure gravity and zero Λ-term) can be written as

$$\exp\big(-I\big) = \prod_A \prod_{i,j,k,l,m} \Bigg(1 + \frac{1}{5 \times 24 \times l_P^2}\, \varepsilon_{Aijklm}\, \mathrm{tr}\, \gamma^5 \times \times \Omega_{Ami}\Omega_{Aij}\Omega_{Ajm}\hat{e}_{Amk}\hat{e}_{Aml}\Bigg) \, . \tag{2.45}$$

The expansion (2.45) is called further as high temperature expansion. It is well known that the analogous representation for the $\exp\big(-I\big)$ is true in the lattice Yang–Mills theory in the limit of large coupling constant. From such representation the significant phenomenon of colour confinement follows. Originally the phenomenon of colour confinement has been obtained analytically with the help of high temperature expansion (with the help of representation of the tipe (2.45)) by Wilson, and then numerous computer simulations confirmed this conclusion. Since the situations concerning high temperature expansion in both theories are closely analogous, we make the conclusion that in the region of variables (2.44) also take place colour confinement. Introduce the following notations:

$$C = \{a_{i_0}a_{i_1},\, a_{i_1}a_{i_2}, \ldots , a_{i_r}a_{i_0}\}$$

is a closed contour ore a one dimensional subcomplex with zero boundary;

$$W\,(C) = \langle\, \mathrm{tr}\,(\Omega_{i_0 i_1}\Omega_{i_1 i_2} \ldots \Omega_{i_r i_0})\rangle_1$$

is Wilson loop correlator which in our case is calculated in the theory of pure gravity with zero Λ-term in the region of variables restricted by inequalities (2.44); σ_C is a two dimensional subcomplex with boundary $\partial\,\sigma = C$; $n_C(\sigma)$ is the number of triangles containing in σ and

$$n_C = \min_\sigma \{n_C(\sigma)\} \, .$$

Then the simple calculations give the following estimation:

$$W\,(C) \sim \exp\left(-n_C \mu\, \ln l_1^{-1}\right) \, . \tag{2.46}$$

Here μ is a number which does not depend on contour C and parameter l_1.

Let us emphasize that in the case of discrete quantum gravity the role of colour gauge group play the group (2.14). Thus only singlet (i.e. scalar, but not spinor, vector and so on) fields with respect to the group (2.14) have quasiparticle excitations in the region (2.44), i.e. on the early stages of universe development. This conclusion partially justifies the use only scalar fields in numerous works in which the dynamics of early universe is investigated. But in contrast to the Yang–Mills theory in expanding universe the phase transition occurs to deconfinement phase (formally in the region (2.43)). In this phase the dynamics becomes quasiclassical.

Let us now show that the modes of quantized fields in the quasiclassical continual phase have essentially noncompact packing in momentum space. This important conclusion follows from high temperature expansion and the most general properties of spectrum of elliptic operators.

We illustrate the effect in Appendix A on the example of the spectrum of one dimensional discrete Laplace operator on random lattice on a cycle. In the cases of 3 and 4 vertexes the problem is solved exactly and we see that in the case when the total length of the cycle is fixed but the distances between some vertexes tend to zero some of eigenfunctions of the operator tend to infinity as inverse degrees of the small distances between the corresponding vertexes.

Let us make the estimation of modes packing in our theory in 3-dimensional space. We keep in mind the scalar field since the spinor structure does not affect significantly for the estimation.

Firstly, we write out the trivial formula for for the volume in momentum space occupied by N modes placed in the flat volume V and densely packed in momentum space:

$$\Omega = (2\pi)^3 \frac{N}{V} \, . \tag{2.47}$$

Now, one must take into account the fact that in confinement phase all correlators of fundamental fields drop exponentially with space separation. This means that the fields at nearest regions of space volume are not correlated. The same conclusion remains true at initial times in quasiclassical phase. Therefore let us divide a macroscopic volume V with the total number of degrees of freedom (ore the number of modes) N into $\mathcal{N}$ subvolumes v_i in each of which contains n_i degrees of freedom. Thus

$$\sum_{i=1}^{\mathcal{N}} n_i = N \, , \qquad \sum_{i=1}^{\mathcal{N}} v_i = V \, , \tag{2.48}$$

and

$$\omega_i = (2\pi)^3 \frac{n_i}{v_i} \tag{2.49}$$

is the minimal possible volume in momentum space occupied by n_i modes placed in the flat volume v_i. Now instead of the quantity (2.47) one must consider the fol-

lowing quantity:

$$\tilde{\Omega} = \frac{(2\pi)^3}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \frac{n_i}{v_i} \,. \tag{2.50}$$

Indeed, the minimum of quantity (2.50) subjected to the constraints (2.48) is equal to (2.47).

But in considered theory the volumes v_i are variable quantities. Therefore one must introduce the measure on the manifold of volumes $\{v_i\}$. The simplest measure agreed with fundamental measure (2.34) looks like the following:

$$\mathrm{d}\,\mu = \frac{(\mathcal{N}-1)!}{V^{\mathcal{N}-1}} \delta\left(V - \sum_{i=1}^{\mathcal{N}} v_i\right) \prod_{i=1}^{\mathcal{N}} \mathrm{d}\,v_i \,, \quad v_i > 0 \,,$$
$$\int \mathrm{d}\,\mu = 1 \,. \tag{2.51}$$

Hence instead of (2.50) the more physically sensible quantity is

$$\langle\tilde{\Omega}\rangle \equiv \int \tilde{\Omega}\,\mathrm{d}\,\mu = (2\pi)^3 \frac{\mathcal{N}-1}{V\,\mathcal{N}} \sum_{i=1}^{\mathcal{N}} n_i \int_{v_i \ll V} \frac{\mathrm{d}\,v_i}{v_i} =$$
$$= (2\pi)^3 \frac{N}{V} \int_{v_i \ll V} \frac{\mathrm{d}\,v_i}{v_i} \,. \tag{2.52}$$

The last equality is obtained taking into account the first constraint of (2.48) and the relation $\mathcal{N} \gg 1$.

The comparison of Eqs. (2.47) and (2.52) shows that taking into account the dynamics of the system leads to the essential expansion of the momentum space volume occupied by quantum field modes. This expansion factor is

$$\varkappa_1 = \int_{v_i \ll V} \frac{\mathrm{d}\,v_i}{v_i} = 3 \ln \frac{a_1}{a_0} = 3 \ln \xi_0 \,. \tag{2.53}$$

Here a_0 is some minimal dimension of the theory and $a_1 \sim V^{1/3}$.

Now there is a need to make a kind of renorm-group. Let n be the number of steps of renorm-group and

$$\xi_s = \frac{a_{s+1}}{a_s} = \xi \gg 1 \,, \quad s = 1, \ldots, n \,, \tag{2.54}$$

and $a_{n+1} = a$ is the radius of universe. Thus $\xi^n = a/a_0$. Let us take

$$n = \frac{1}{\lambda} \ln \frac{a}{a_0} \gg 1 \,, \quad \lambda \gg 1 \,. \tag{2.55}$$

Using Eqs. (2.53)–(2.55) it is easy to see that the expansion factor of momentum space volume occupied by modes after n steps is

$$\varkappa_n = \prod_{s=1}^{n} (3 \ln \xi_s) = (3 \ln \xi)^n = \left(\frac{a}{a_0}\right)^{(\ln 3\lambda)/\lambda} \,. \tag{2.56}$$

The value of right hand side of Eq. (2.56) can bee very large (many orders) in magnitude.

It follows from the presented analysis, that the continual quantum gravity arising from the discrete quantum gravity (if it exists) possess very unusual properties. The description of possible such theory and some consequence is performed in the subsequent sections.

III. METHOD OF DYNAMIC QUANTIZATION

Specific results of the application of the Dynamic quantization method to the two-dimensional theories [4, 17], obtained by explicit constructions and direct calculations, justify the abstract assumptions and axioms on which this method is based.

We shall explain the ideology and logical scheme of the Dynamic method taking account of the experience in quantizing two-dimensional gravity.

The key point in the quantization of two-dimensional gravity was the construction of a complete set of such operators $\{A_n, B_n, \ldots\}$, designated below as $\{A_N,\, A_N^\dagger\}$, which possess the following properties:

1) The operators A_N and $A_N^\dagger$ are Hermithian conjugates of one another and

$$[\,A_N,\, A_M\,] = 0, \qquad [A_N,\, A_M^\dagger\,] = \delta_{NM} \,. \tag{3.1}$$

2) The set of operators $\{A_N,\, A_N^\dagger\}$ describes all physical dynamical degrees of freedom of a system.

3) Each operator from the set $\{A_N,\, A_N^\dagger\}$ commutes weakly with all first class constraints or with the complete Hamiltonian of the theory.

Quantization is performed directly using the operators $\{A_N,\, A_N^\dagger\}$. It means that the space of physical states is created using the operators $\{A_N^\dagger\}$ from the ground state and all operators are expressed in terms of the operators $\{A_N,\, A_N^\dagger\}$, as well as in terms of the operators describing the gauge degrees of freedom. However, in the theory of two-dimensional gravity the operators $\{A_N,\, A_N^\dagger\}$ were constructed explicitly (i.e., they were expressed explicitly in terms of the fundamental dynamical variables), in more realistic theories this problem is hardly solvable. Therefore, the set of operators $\{A_N,\, A_N^\dagger\}$ with properties 1)–3) must be introduced axiomatically. Conversely, the properties 1)–3) make it possible, in principle, to express the initial variables in terms of the convenient operators $\{A_N,\, A_N^\dagger\}$.

However, in contrast to the two-dimensional theory of gravity, regularization is necessary in real models of gravity. In the Dynamic quantization method, regularization is carried out precisely in terms of the operators $\{A_N,\, A_N^\dagger\}$. As will be shown below, such regularization is natural in generally covariant theories, since it preserves the form of the Heisenberg equations and thereby also the general covariance of the theory.

As a result we have the regularized general covariant theory describing quantum gravity, the main property of which is the finiteness of physical degrees of freedom contained in each finite volume. Moreover, the packing of field modes in momentum space can be made rare. Evidently, the theory of discrete quantum gravity described in Section 2 possess the same properties. Therefore, one can think that the theory of gravity quantized by dynamic quantization method is the continuous limit of discrete quantum gravity.

Let's consider a generally covariant field theory. Let us assume that in this theory the Hamiltonian in the classical limit is an arbitrary linear combination of the first class constraints and there are no the second class constraints.

Let $\{\Phi^{(i)}(x),\, P^{(i)}(x)\}$ be a complete set of fundamental fields of the theory and their canonically conjugate momenta, in terms of which all other physical quantities and fields of the theory are expressed. Here the index (i) enumerates the tipe of fields. For example, for some (i) these can be either 6 spatial components of the metric tensor $g_{ij}(x)$ or the scalar field $\phi(x)$ or the Dirac field $\psi(x)$ etc. The set of fields $\{\Phi^{(i)}(x)\}$ is a complete set of the mutually commuting (at least in formal unregularized theory) fundamental fields of the theory.

Next, to simplify the notation the index i will be omitted. It can be assumed that the variable x includes, besides the spatial coordinates, the index i also.

The construction of a quantum theory by the Dynamic method is based on the following natural assumptions relative to the structure of the space F of the physical states of the theory.

1. *All states of the theory having physical sense are obtained from the ground state $|\,0\,\rangle$ using the creation operators $A^\dagger_N$:*

$$\begin{gathered}|\, n_1,\, N_1; \ldots ;\, n_s,\, N_s\,\rangle = \\ = (n_1! \cdot \ldots \cdot n_s!\,)^{-\frac{1}{2}} \cdot (A^\dagger_{N_1})^{n_1} \cdot \ldots \cdot (A^\dagger_{N_s})^{n_s}\,|\,0\,\rangle\ , \\ A_N\,|\,0\,\rangle = 0\,. \end{gathered} \tag{3.2}$$

States (3.2) form an orthonormal basis of the space F of physical states of the theory.

The numbers $n_1, \ldots, n_s$ assume integer values and are called occupation numbers.

2. *The set of states $\Phi(x)\,|\,n_1, N_1; \ldots ; n_s, N_s\,\rangle$, , where the set of numbers $(n_1, N_1; \ldots ; n_s, N_s)$ is fixed, contains a superposition of* **all** *states of the theory, for which one of the occupation number differs in modulus by one and all other occupation numbers equal to the occupation numbers of state (3.2).*

Here the operators $A^\dagger_N$ and their conjugates A are the generators of Heisenberg algebra. The operators $\{A_N,\, A^\dagger_N\}$, generally speaking, can be bosonic or fermionic. If the creation and annihilation operators follow the Fermi statistics, then the anticommutator are used. For the case of compact spaces which is interesting for us, we can assume without loss of generality that the index N, enumerating the creation and annihilation operators, belongs to a discrete finite-dimensional lattice. A norm can be easily introduced in the space of indexes N.

Since states (3.2) are physical, they satisfy the relations

$$\mathcal{H}_T\,|\, n_1,\, N_1; \ldots ;\, n_s,\, N_s\,\rangle = 0\ , \tag{3.3}$$

where $\mathcal{H}_T$ is the complete Hamiltonian of the theory. We assume that $\mathcal{H}_T = \sum_\Xi v_\Xi \chi_\Xi$, where $\{\chi_\Xi\}$ is the complete set of the first class constraints and $\{v_\Xi\}$ is arbitrary set of Lagrange multipliers.

Equations (3.2) and (3.3) are compatible if and only if the following relations are valid:

$$\begin{gathered}[A_N,\, \chi_\Xi] = \sum_\Pi c_{N\Xi\Pi}\,\chi_\Pi\ , \\ [A^\dagger_N,\, \chi_\Xi] = -\sum_\Pi \chi_\Pi\, c^\dagger_{N\Xi\Pi} = \sum_\Pi \tilde{c}_{N\Xi\Pi}\,\chi_\Pi\,. \end{gathered} \tag{3.4}$$

Since the coefficients $c_{N\Xi\Pi},\ \tilde{c}_{N\Xi\Pi}$ in Eq. (3.4) generally are operators, the arrangement of the multiplies in the right hand sides of Eqs. (3.4) is important.

Let $(A^\dagger_N,\, A_N)$ be a pair of bose or fermi creation and annihilation operators creating or annihilating the state with the wave function $\psi_N(x)$. According to (3.4) we have:

$$\begin{gathered}[A_N,\, \mathcal{H}_T] = \sum_{\Xi,\,\Pi} r_{N\,\Xi\,\Pi} v_\Xi\, \chi_\Pi \longleftrightarrow \\ \longleftrightarrow [A^\dagger_N,\, \mathcal{H}_T] = -\sum_{\Xi,\,\Pi} \chi_\Pi\, v^*_\Xi r^\dagger_{N\,\Xi\,\Pi}\,. \end{gathered} \tag{3.5}$$

Let an arbitrary operator Φ be represented as a normal ordered power series in operators $(A^\dagger_N,\, A_N)$:

$$\Phi = \Phi' + \phi_N A_N + A^\dagger_N \tilde{\phi}_N\,. \tag{3.6}$$

By definition, here the operator Φ' does not depend on the operators $(A^\dagger_N,\, A_N)$:

$$[\Phi',\, A^\dagger_N] = [\Phi',\, A_N] = 0\,. \tag{3.7}$$

It follows from Eqs. (3.5)–(3.7) that

$$\begin{gathered}[\Phi,\, \mathcal{H}_T] = [\Phi',\, \mathcal{H}'_T] + \\ + \sum_\Xi (q_\Xi \chi_\Xi + \chi_\Xi \tilde{q}_\Xi) + (p_N A_N + A^\dagger_N \tilde{p}_N)\,. \end{gathered} \tag{3.8}$$

Here the total Hamiltonian $\mathcal{H}_T$ is represented according to (3.6), so that $\mathcal{H}'_T$ does not depend on the operators $(A^\dagger_N,\, A_N)$. To verify Eq. (3.8) let's write out the following chain of equalities:

$$\begin{gathered}[\Phi,\, \mathcal{H}_T] = [\Phi',\, \mathcal{H}_T] + \Big(\phi_N [A_N,\, \mathcal{H}_T] + [A^\dagger_N,\, \mathcal{H}_T]\,\tilde{\phi}_N\Big) + \\ + \Big([\phi_N,\, \mathcal{H}_T]\, A_N + A^\dagger_N [\tilde{\phi}_N,\, \mathcal{H}_T]\Big)\ . \end{gathered} \tag{3.9}$$

As a consequence of Eqs. (3.5) the second item in the right hand side of Eq. (3.9) has the same structure as the second item in the right hand side of Eq. (3.8). Evidently, the last items in the right hand side of Eq. (3.9) has the same structure as the last items in the right hand side of Eq. (3.8). Now let's write out the following identity:

$$[\Phi',\, \mathcal{H}_T] \equiv [\Phi',\, \mathcal{H}'_T] + [\Phi',\, \mathcal{H}_T - \mathcal{H}'_T]\,. \tag{3.10}$$

By definition

$$\mathcal{H}_T - \mathcal{H}'_T = h_N\, A_N + A^\dagger_N\, \tilde{h}_N\,. \tag{3.11}$$

It follows from (3.7) and (3.11) that

$$[\Phi',\, \mathcal{H}_T - \mathcal{H}'_T] = [\Phi',\, h_N]\, A_N + A^\dagger_N\, [\Phi',\, \tilde{h}_N]\,. \tag{3.12}$$

Combining Eqs. (3.9), (3.10) and (3.12) we come to the Eq. (3.8).

Now let's impose an additional pair of second class constraints

$$A_N = 0\,, \qquad A^\dagger_N = 0\,. \tag{3.13}$$

By definition under the constraints (3.13) any operator Φ is reduced to the operator Φ' in (3.6). The Dirac bracket arising under the constraints (3.13) is defined according to the following equality:

$$[\Phi,\, \mathcal{F}]^* \equiv [\Phi',\, \mathcal{F}']\,. \tag{3.14}$$

The remarkable property of the considered theory is the fact that

$$[\Phi,\, \mathcal{H}_T]^* \approx [\Phi,\, \mathcal{H}_T]\,. \tag{3.15}$$

Here the approximate equality means that after the imposition of all first and second class constraints the operators in the both sides of Eq. (3.15) coincide, that is the weak equality (3.15) reduces to the strong one. Relation (3.15) follows immediately from Eqs. (3.5) and (3.14). Eq. (3.15) means that the Heisenberg equation

$$i\dot{\Phi} = [\Phi,\, \mathcal{H}_T]^*$$

for any field in reduced theory coincides weakly with corresponding Heisenberg equation in nonreduced theory. Evidently, this remarkable conclusion retains true under imposition of any number of pairs of the second class constraints of type (3.13) [20].

The above-stated bring to the following idea of ultraviolet regularization of quantum theory of gravity. Let a local field $\Phi(x)$ create and annihilate particles in the states with wave functions $\{\phi_N(x)\}$ by creation and annihilation operators $\{A^\dagger_N,\, A_N\}$ (for simplicity the field Φ is assumed to be real). The physical space of states is invariant relative to the action of creation and annihilation operators. Therefore there is the possibility of imposing the second class constraints of the type (3.13) for any number of pairs of these operators without changing Heisenberg equations of motion. Let the high-frequency (in some sense) wave functions $\{\phi_N(x)\}_{|N|>N_0}$ have the value of index $|N| > N_0$. The ultraviolet regularization of the theory is performed by imposing the constraints of the type (3.13) for all $|N| > N_0$. It is very important that under the constraints the regularized equations of motion and first class constraints preserve their canonical form. Hence the equations of regularized theory are general covariant, i.e. they conserve their form under the general coordinate transformations and local frame transformations.

Since unregularized theory of quantum gravity is mathematically meaningless, so it seems correct the direct definition of regularized theory by means of introduction of natural axioms.

Axiom 1. *All states of the theory which are physically meaningful are obtained from the ground state* $|\,0\,\rangle$ *using the creation operators* $A^\dagger_N$ with $|\,N\,| < N_0$:

$$\begin{gathered} |\,n_1,\, N_1; \ldots ;\, n_s,\, N_s\,\rangle = \\ = (n_1! \cdot \ldots \cdot n_s!\,)^{-\frac{1}{2}} \cdot (A^\dagger_{N_1})^{n_1} \cdot \ldots \cdot (A^\dagger_{N_s})^{n_s}\,|\,0\,\rangle\,, \\ A_N\,|\,0\,\rangle = 0\,. \end{gathered} \tag{3.16}$$

States (3.16) form an orthonormal basis of the space F *of physical states of the theory.*

Axiom 2. *The dynamical variables* $\Phi(x)$ *transfer state (3.16) with fixed values of numbers* $(n_1,\, N_1; \ldots ;\, n_s,\, N_s)$ *into a superposition of the states of the theory of form (3.16), containing all states in which one of the occupation numbers is different in modulus by one and all other occupation numbers are identical to those of state (3.16).*

Axiom 3. *The equations of motion and constraints for the physical fields* $\{\Phi(x),\, \mathcal{P}(x)\}$ *have the same form, to within the arrangement of the operators, as the corresponding classical equations and constraints.*

Further we suppose that the momentum variables $\mathcal{P}(x)$ are expressed through the fundamental field variables $\Phi(x)$ and their time derivatives $\dot{\Phi}(x)$, so that the Lagrange equations instead of Hamilton equations are used.

Let's assume, further, that the ground state $|\,0\,\rangle$ is a coherent state with respect to the gauge degrees of freedom. It means that the quantum fluctuations of the gauge degrees of freedom are not significant and their dynamics in fact is classical.

Let's emphasize that this assumption is related with the fact of noncompactness of the gauge group. (Since the group of general linear transformations is noncompact, so the gauge group in the theory of gravity is noncompact.) The quasiclassical charecter of dynamics of gauge degrees of freedom seems true only for noncompact gauge groups. On the contrary, the motion in compact

gauge group (such as in Yang-Mills theory) can not be regarded as classical.

Let's consider, for example, the quantized electrodynamic field in noncovariant Coulomb gauge. In this gauge only the degrees of freedom describing photons fluctuate, but the gauge (longitudinal) degrees of freedom are defined unambiguously through the electric current. Thus, the gauge degrees of freedom in QED does not fluctuate, effectively they are classical. On the other hand, in high-temperature confinement phase in QED on a lattice the high-temperature expansion is valid. In this case the gauge degrees of freedom can not be regarded as classical. So our assumption about classical behavior of gauge degrees of freedom in quantum gravity is equivalent to the assumption that quantum gravity is in noncompact phase.

Consider any fundamental field:

$$\Phi(x) = \Phi_{(cl)}(x) + \sum_{|N|<N_0} [\,\phi_N(x)\,A_N + \phi^*_N(x)\,A^\dagger_N\,] + \ldots. \tag{3.17}$$

On the right-hand side of Eq. (3.17) *all functions* $\Phi_{(cl)}(x)$, $\phi_N(x)$, *and so on are c-number functions.*

This follows from the assumption about the quasiclassical character of the dynamics of gauge degrees of freedom.

Now we can supplement our system of axioms by the following supposition: field (3.17) is used in axioms 1-3. The fields $\Phi_{(cl)}(x)$, $\phi_N(x)$, $\psi_N(x)$, and so on satisfy certain equations which can be obtained uniquely from the Lagrange equations of motion, if the expansion of the field $\Phi(x)$ in form (3.17) is substituted into them and then, after normal ordering of the operators $\{\,A_N,\,A^\dagger_N\,\}$, the coefficients of the various powers of the generators of the Heisenberg algebra $\{\,A_N,\,A^\dagger_N\,\}$ are equated to zero. As a result of the indicated normal ordering, the relations arise between the higher order coefficient functions and the lower order coefficient functions in expansion (3.17). We obtain an infinite chain of equations for the coefficient functions $\{\,\Phi^{(cl)}(x),\,\phi_N(x),\,\psi_N(x),\,\ldots\}$. The latter conjecture can be introduced with the aid of the following axiom, replacing axiom 3.

Axiom 3′. *The equations of motion for the quantized fields* (3.17), *up to the ordering of the quantized fields, have the same form as the corresponding classical equations of motion.*

IV. DYNAMIC QUANTIZATION OF GRAVITY

We shall now apply the quantization scheme developed above to the theory of gravity.

Let's consider the theory of gravity with a Λ term which is coupled minimally with the Dirac field. The action of such a theory has the form

$$S = -\frac{1}{l_P^2}\int d^4\,x\,\sqrt{-g}\;(R+2\Lambda)+ \\ +\int d^4\,x\,\sqrt{-g}\,\left\{\frac{i}{2}\,e^\mu_a\Big(\overline{\psi}\gamma^a\,D_\mu\psi - \overline{D_\mu\psi}\,\gamma^a\,\psi\Big) - m\overline{\psi}\psi\right\}. \tag{4.1}$$

Here $\{e^\mu_a\}$ is an orthonormalized basis, $g_{\mu\nu}$ is the metric tensor, and $\eta_{ab} = diag(1,\,-1,\,-1,\,-1)$ so that

$$g_{\mu\nu}\,e^\mu_a\,e^\nu_b = \eta_{ab}, \qquad R = e^\mu_a\,e^\nu_b\,R^{ab}_{\mu\nu}\,,$$

the 2-form of the curvature is given by

$$d\omega^{ab} + \omega^a_c \wedge \omega^{cb} = \frac{1}{2}\,R^{ab}_{\mu\nu}\,dx^\mu \wedge dx^\nu\,,$$

where the 1-form $\omega^a_b = \omega^a_{b\mu}\,dx^\mu$ is the connection in the orthonormal basis $\{e^\mu_a\}$. The spinor covariant derivative is given by the formula

$$D_\mu\psi = \left(\frac{\partial}{\partial x^\mu} + \frac{1}{2}\,\omega_{ab\mu}\,\sigma^{ab}\right)\psi\,, \quad \sigma^{ab} = \frac{1}{4}\,[\gamma^a,\,\gamma^b]\,,$$

γ^a are the Dirac matrices:

$$\gamma^a\,\gamma^b + \gamma^b\,\gamma^a = 2\,\eta^{ab}\,.$$

Let's write out the equations of motion for system (4.1).

The variation of action (4.1) relative to the connection gives the equation

$$\nabla_\mu\,e^a_\nu - \nabla_\nu\,e^a_\mu = -\frac{1}{4}l_P^2\,\varepsilon^a{}_{bcd}\,e^b_\mu e^c_\nu\,\overline{\psi}\gamma^5\gamma^d\,\psi \equiv T^a_{\mu\nu}\,. \tag{4.2}$$

In deriving the last equation, we employed the equality

$$\gamma^a\,\sigma^{bc} + \sigma^{bc}\,\gamma^a = -i\varepsilon^{abcd}\,\gamma^5\,\gamma_d\,. \tag{4.3}$$

Here ε_{abcd} is the absolutely antisymmetric tensor, and $\varepsilon_{0123} = 1$. The right-hand side of Eq. (4.2) is the torsion tensor.

We note that torsion (4.2) possesses the property

$$T^\nu_{\mu\nu} \equiv e^\nu_a\,T^a_{\mu\nu} \equiv 0\,. \tag{4.4}$$

Consequently, even though torsion exists in the considered theory, the torsion tensor is not present in the Dirac equation:

$$\big(i\,e^\mu_a\,\gamma^a\,D_\mu - m\big)\,\psi = 0\,. \tag{4.5}$$

The variation of action (4.1) relative to the orthonormal basis gives the Einstein equation, which we write in the form

$$R_{\mu\nu} + \Lambda\,g_{\mu\nu} = \frac{1}{2}l_P^2\left\{\frac{i}{2}\Big(\overline{\psi}\,\gamma^c\,e_{c(\mu}D_{\nu)}\psi - \\ - e_{c(\mu}\overline{D_{\nu)}\psi}\,\gamma^c\,\psi\Big) - \frac{1}{2}m\,\overline{\psi}\psi\,g_{\mu\nu}\right\}. \tag{4.6}$$

Here the expression in braces is $(T_{\mu\nu} - 1/2\, g_{\mu\nu}\, T)$, where $T_{\mu\nu}$ is the energy-momentum tensor on the mass shell (i.e., taking account of the equations of motion of matter — in our case, the Dirac equation (4.4)).

Equations (4.2), (4.3)–(4.6), together with the relations

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu\,, \quad e^\mu_a\, e^b_\mu = \delta^b_a$$

form a complete system of classical equations of motion and constraints for system (4.1).

We now represent the fields as the sum of classical and quantum components:

$$g_{\mu\nu} = g_{(cl)\mu\nu} + h_{\mu\nu}\ \ ,\ \ e^a_\mu = e^a_{(cl)\mu} + f^a_\mu\,. \tag{4.7}$$

Assume that the fermion field has no classical component, so that

$$\psi(x) = \sum_{|N|<N_F} \left(a_N\, \psi^{(+)}_N(x) + b^\dagger_N\, \psi^{(-)}_N(x) \right) + \ldots\ \ , \tag{4.8}$$

where the Fermi creation and annihilation operators satisfy the following anticommutation relations (as usual, only the nonzero relations are written out):

$$\begin{gathered} \{\, a_M\,,\, a^\dagger_N\,\} = \{\, b_M\,,\, b^\dagger_N\,\} = \delta_{M,N}\,, \\ a_N|0\rangle = b_N|0\rangle = 0\,. \end{gathered} \tag{4.9}$$

The complete orthonormal set of fermion modes $\left\{\psi^{(\pm)}_N(x)\right\}$ can be naturally determined as follows. Denote by $\Sigma^{(3)}$ the spacelike hypersurface, defined by the equation $t = \text{Const}$, and by $\Sigma^{(3)}_0$ the hypersurface at $t = t_0$. Let the metric in space-time be given by means of the tensor $g_{\mu\nu}$. This metric induces a metric on $\Sigma^{(3)}_0$, which in the local coordinates x^i, $i = 1, 2, 3$, is represented by the metric tensor ${}^3g_{ij}$. Using the equations

$${}^3g_{ij,k} = \gamma^l_{ik}\, {}^3g_{lj} + \gamma^l_{jk}\, {}^3g_{il}\,, \quad \gamma^k_{ij} = \gamma^k_{ji}\,,$$

$${}^3g_{ij} = -\sum_{\alpha=1}^{3} {}^3e^\alpha_i\, {}^3e^\alpha_j\,, \quad {}^3e^\alpha_i\, {}^3e^i_\beta = \delta_{\alpha\beta}\,,$$

$$\partial_i\, {}^3e^i_\alpha + \gamma^j_{ki}\, {}^3e^k_\alpha + {}^3\omega_{\alpha\beta i}\, {}^3e^j_\beta = 0\,, \quad {}^3\omega_{\alpha\beta i} = -{}^3\omega_{\beta\alpha i}$$

the connection (without torsion) γ^i_{jk} in local coordinates and a spin connection ${}^3\omega_{\alpha\beta i}$ are determined on $\Sigma^{(3)}_0$. For a Dirac single-particle Hamiltonian we have:

$$\mathcal{H}_D = -i\, {}^3e^i_\alpha\, \alpha^\alpha\, (\partial_i + \frac{1}{2}\, {}^3\omega_{\beta\gamma i}\, \frac{1}{4}\, [\alpha^\beta,\, \alpha^\gamma\,]\,) + m\, \gamma^0\,,$$

$$\alpha^\beta = \gamma^0\, \gamma^\beta$$

It is easy to check that in the metric

$$\langle\, \psi_M,\, \psi_N\,\rangle = \int_{\Sigma^{(3)}_0} d^3x\, \sqrt{-{}^3g}\, \psi^\dagger_M\, \psi_N \tag{4.10}$$

the operator $\mathcal{H}_D$ is self-conjugated. Consequently, the solution of the problem for the eigenvalues on $\Sigma^{(3)}_0$

$$\mathcal{H}^{(0)}_D\, \psi^{(\pm)}_N(x) = \pm\varepsilon_N\, \psi^{(\pm)}_N(x)\,, \quad \varepsilon_N > 0 \tag{4.11}$$

has a complete set of orthonormal modes in metric (4.10). The index (0) everywhere means that in the corresponding quantity the fields are taken in the zero approximation with respect to quantum fluctuations.

Note that a one-to-one relation can be established between the positive- and negative-frequency modes by means of the equation

$$\gamma^0\gamma^5\, \psi^{(+)}_M = \psi^{(-)}_M$$

We call the attention to the fact that the scalar product

$$(\psi_M,\, \psi_N) = \int_{\Sigma^{(3)}} d^3x\, \sqrt{-g^{(0)}}\, \psi^\dagger_M\, \psi_N \tag{4.12}$$

is not always the same as the scalar product (4.10). These scalar products coincide, if the path function $N = 1$, which happens, for example, for the metric

$$g^{(0)}_{0i} = 0\,, \quad g^{(0)}_{00} = 1\,.$$

The scalar product (4.12) has the advantage over the scalar product (4.10) that if the modes $\{\psi^{(\pm)}_N(x)\}$ satisfy the Dirac equation in the zero approximation with respect to quantum fluctuations (which, according to the exposition below, does indeed happen), then the scalar product (4.12) is conserved in time.

The field $h_{\mu\nu}$ in Eq. (4.7) can be expanded as follows:

$$\begin{aligned} h_{\mu\nu} = {} & l_P \sum_{|N|<N_0} (h_{N\ \mu\nu} c_N + h^*_{N\ \mu\nu} c^\dagger_N) + \\ & + l_P^2 \Big\{ \sum_{|N_1|,|N_2|<N_0} (h_{N_1 N_2\ \mu\nu} c_{N_1} c_{N_2} + h^*_{N_1 N_2\ \mu\nu} c^\dagger_{N_1} c^\dagger_{N_2} + \\ & + h_{N_1|N_2\ \mu\nu} c^\dagger_{N_1} c_{N_2}) + \\ & + \sum_{|N_1|,\,|N_2|<N_F} (\, h^{F(++)}_{N_1\, N_2\, \mu\nu}\, a^\dagger_{N_1}\, a_{N_2} + h^{F(--)}_{N_2\, N_1\, \mu\nu}\, b^\dagger_{N_1}\, b_{N_2} + \\ & + h^{F(+-)}_{N_1\, N_2\, \mu\nu}\, a^\dagger_{N_1}\, b^\dagger_{N_2} + h^{F(+-)*}_{N_1\, N_2\, \mu\nu}\, b_{N_2}\, a_{N_1}) \Big\} + \ldots \end{aligned} \tag{4.13}$$

In Eqs. (4.7), (4.8), and (4.13) the c-number coefficient fields $\psi^{(\pm)}_N$, $g_{(cl)\,\mu\nu}$, $h_{N\,\mu\nu}$ and so on can be expanded in powers of the Planck scale, for example

$$g_{(cl)\,\mu\nu} = g^{(0)}_{\mu\nu} + l_p^2\, g^{(2)}_{(cl)\,\mu\nu} + \ldots$$

Since fields (4.13) are real, we have

$$\begin{gathered} h_{N_1 N_2\ \mu\nu} = h_{N_2 N_1\ \mu\nu}\,,\ h^*_{N_1|N_2\ \mu\nu} = h_{N_2|N_1\ \mu\nu}\,, \\ h^{F(++)*}_{N_2\, N_1\, \mu\nu} = h^{F(++)}_{N_1\, N_2\, \mu\nu}\,,\ h^{F(--)*}_{N_2\, N_1\, \mu\nu} = h^{F(--)}_{N_1\, N_2\, \mu\nu} \end{gathered} \tag{4.14}$$

The operators $\{c_N,\, c_N^{+}\}$ satisfy the Bose commutation relations (4.4). A method for choosing the set of functions $\{h_{N\,\mu\nu}\}$ will be discussed below.

According to the dynamic quantization scheme, we must substitute fields (4.7)–(4.8) and (4.13) into Eqs. (4.2) and (4.5)–(4.6), after which the operators $\{A_N,\, A_N^{\dagger}\}$ must be normal-ordered and all coefficients of the various powers of these operators and the Planck scale must be set equal to zero.

Thus, we obtain the first of these equations:

$$\nabla^{(0)}_{\mu}\, e^{(0)\,a}_{\nu} - \nabla^{(0)}_{\nu}\, e^{(0)\,a}_{\mu} = 0\,, \quad R^{(0)}_{\mu\nu} + \Lambda\, g^{(0)}_{\mu\nu} = 0 \qquad (4.15)$$

Here and below all raising and lowering of indices are done with the tensors $g^{(0)}_{\mu\nu}$ and $g^{(0)\mu\nu}$. Thus, in the lowest approximation the fields satisfy the classical equations of motion. In the zeroth approximation we also have a series of equations for the fermion modes:

$$\left(i\, e^{(0)\,\mu}_{a}\, \gamma^{a}\, D^{(0)}_{\mu} - m\right) \psi^{(0)(\pm)}_{N} = 0 \qquad (4.16)$$

We now introduce the notation

$$K^{(0)\lambda\rho}_{\mu\nu} = \Big[-\frac{1}{2}\nabla^{(0)}_{\sigma}\nabla^{(0)\sigma}\,\delta^{\lambda}_{\mu}\,\delta^{\rho}_{\nu} - R^{(0)\lambda\ \rho}_{\ \ \mu\ \nu} + R^{(0)\rho}_{\nu}\,\delta^{\lambda}_{\mu} + \\ +\nabla^{(0)}_{\mu}\left(\nabla^{(0)\lambda}\,\delta^{\rho}_{\nu} - \frac{1}{2}\,\nabla^{(0)}_{\nu}\, g^{(0)\lambda\rho}\right)\Big] + \\ +[\,\mu \longleftrightarrow \nu] + 2\,\Lambda\,\delta^{\lambda}_{(\mu}\delta^{\rho}_{\nu)}\,, \qquad (4.17)$$

$$R^{(0)(2)}_{\mu\nu}(h,\, h') = \frac{1}{2}\Big[\, R^{(0)(2)}_{\mu\nu}(h+h',\, h+h') - \\ -R^{(0)(2)}_{\mu\nu}(h,\, h) - R^{(0)(2)}_{\mu\nu}(h',\, h')\Big] \qquad (4.18)$$

It is easily checked that

$$\frac{1}{2}\, K^{(0)\,\lambda\rho}_{\mu\nu} = \frac{\delta\,(R_{\mu\nu} + \Lambda\, g_{\mu\nu})}{\delta\, g_{\lambda\rho}}\bigg|_{g_{\mu\nu}=g^{(0)}_{\mu\nu}}\,,$$

where $R^{(0)(2)}_{\mu\nu}(h,\, h)$ is a quadratic form of the tensor field $h_{\lambda\rho}$ which can be constructed in terms of the second variation of $R_{\mu\nu}$ relative to the metric tensor at the point $g^{(0)}_{\mu\nu}$. Let's write out the complete form:

$$R^{(0)(2)}_{\mu\nu}(h,h) = \frac{1}{2}\,(h^{\rho}_{\lambda}\, h^{\lambda}_{\rho;\mu}\,)_{;\nu} - \\ -\frac{1}{2}\,[\, h^{\lambda}_{\sigma}\,(h^{\sigma}_{\mu;\nu} + h^{\sigma}_{\nu;\mu} - h^{;\sigma}_{\mu\nu}\,)\,]_{;\lambda} + \\ +\frac{1}{4}\, h^{\lambda}_{\lambda;\rho}\,(h^{\rho}_{\mu;\nu} + h^{\rho}_{\nu;\mu} - h^{;\rho}_{\mu\nu}\,) - \\ -\frac{1}{4}\,(h^{\lambda}_{\rho;\nu} + h^{\lambda}_{\nu;\rho} - h^{;\lambda}_{\nu\rho}\,)\,(h^{\rho}_{\mu;\lambda} + h^{\rho}_{\lambda;\mu} - h^{;\rho}_{\mu\lambda}\,)$$

Thus, $R^{(0)(2)}_{\mu\nu}(h,\, h')$ is a symmetric bilinear form with respect to its arguments $h_{\mu\nu}$ and $h'_{\lambda\rho}$, which in what follows are operator fields (4.13). Thus, here the problem of ordering the operator fields to lowest order has been solved.

Now we can write out the following relations, which follow from the exact quantum equations with the expansion indicated above. To first order in l_P we have

$$\frac{1}{2}\, K^{(0)\,\lambda\rho}_{\mu\nu}\, h_{N\,\lambda\rho} = 0\,. \qquad (4.19)$$

We note that, using Eqs. (4.15), the operator (4.17) vanishes on the quantity $(\xi_{\mu\,;\nu} + \xi_{\nu\,;\mu})$. Consequently, the value of the operator (4.17) on the fields $h_{\mu\nu}$ and

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu\,;\nu} + \xi_{\nu\,;\mu} \qquad (4.20)$$

coincide for any vector field ξ_{μ}. This fact is a consequence of the gauge invariance of the theory. Using the indicated gauge invariance, any solution of Eq. (4.19) can be put into the form

$$\nabla^{(0)}_{\nu}\, h^{\nu}_{\mu} - \frac{1}{2}\,\nabla^{(0)}_{\mu}\, h^{\nu}_{\nu} = 0\,. \qquad (4.21)$$

In what follows, we shall assume that the field satisfies the gauge condition (4.21), which is convenient in a number of problems. It is obvious that taking account of the gauge condition (4.21) the terms in round brackets in operator (4.17) vanishes.

To clarify the question of the normalization of the gravitational modes, we shall employ the following technique. The equation of motion (4.19) can be obtained with the help of the action

$$S^{(2)} = \int d^4x\, \sqrt{-g^{(0)}}\, h^{\mu\nu}\, K^{(0)\,\lambda\rho}_{\mu\nu}\, h_{\lambda\rho}\,. \qquad (4.22)$$

Hence follows the canonically-conjugate momentum for the field $h_{\mu\nu}$ and the simultaneous commutation relations:

$$\pi^{\mu\nu} = \sqrt{-g^{(0)}}\,\nabla^{(0)0}\, h^{\mu\nu}\,, \\ [h_{\mu\nu}(x),\, \pi^{\lambda\rho}(y)\,] = i\,\delta^{\lambda}_{(\mu}\,\delta^{\rho}_{\nu)}\,\delta^{(3)}(x-y)\,. \qquad (4.23)$$

Evidently, in Eq. (4.23) the fields are free of constraints (4.21). Let's represent the field $h_{\mu\nu}$ in the form (compare with the first term in Eq. (4.13))

$$h_{\mu\nu}(x) = \sum_{N}\left(h_{N\,\mu\nu}(x)\, c_N + h^{*}_{N\,\mu\nu}(x)\, c^{\dagger}_{N}\right). \qquad (4.24)$$

The set of operators $\{c_N,\, c^{\dagger}_{N}\}$ form a Heisenberg algebra, and the functions $\{h_{N\,\mu\nu}\}$ satisfy Eq. (4.19). Equations (4.23) and (4.24) lead to the following relations reflecting the orthonormal nature of the set of the modes:

$$i\int_{\Sigma^{(3)}} d^3x\, \sqrt{-g^{(0)}}\,\left[h^{\mu\nu *}_{M}\,\nabla^{(0)\,0}\, h_{N\,\mu\nu} - \\ -(\nabla^{(0)\,0}\, h^{\mu\nu\,*}_{M})\, h_{N\,\mu\nu}\,\right] = \delta_{M,\,N}\,. \qquad (4.25)$$

In the latter equations the integration extends over any spacelike hypersurface $\Sigma^{(3)}$. As a result of Eqs. (4.19), integrals (4.25) indeed do not depend on the hypersurface. It is natural to assume that the gravitational modes satisfy conditions (4.25). The significance of Eqs. (4.25) is that the normalization of the coefficient functions in expansion (4.13) is given with its help.

In the second order in l_P, we obtain the following equations:

$$\frac{1}{2}\,K^{(0)\,\lambda\rho}_{\mu\nu}\,h_{N_1 N_2\,\lambda\rho} = -R^{(0)(2)}_{\mu\nu}\,(h_{N_1},\,h_{N_2})\;, \tag{4.26}$$

$$\frac{1}{2}\,K^{(0)\,\lambda\rho}_{\mu\nu}\,h_{N_1|N_2\,\lambda\rho} = -2\,R^{(0)(2)}_{\mu\nu}\,(h^*_{N_1},\,h_{N_2})\;, \tag{4.27}$$

$$\begin{aligned}\frac{1}{2}\,K^{(0)\,\lambda\rho}_{\mu\nu}\,h^{F(\pm\pm)}_{N_1\,N_2\,\lambda\rho} = \pm\frac{i}{4}\Big(&\overline{\psi}^{(0)(\pm)}_{N_1}\,\gamma^c\,e^{(0)}_{c(\mu}\,D^{(0)}_{\nu)}\,\psi^{(0)(\pm)}_{N_2} - \\ &-e^{(0)}_{c(\mu}\overline{D^{(0)}_{\nu)}\,\psi^{(0)(\pm)}_{N_1}}\,\gamma^c\,\psi^{(0)(\pm)}_{N_2}\Big)\,,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\frac{1}{2}\,K^{(0)\,\lambda\rho}_{\mu\nu}\,h^{F(+-)}_{N_1\,N_2\,\lambda\rho} = \frac{i}{4}\Big(&\overline{\psi}^{(0)(+)}_{N_1}\,\gamma^c\,e^{(0)}_{c(\mu}\,D^{(0)}_{\nu)}\,\psi^{(0)(-)}_{N_2} - \\ &-e^{(0)}_{c(\mu}\overline{D^{(0)}_{\nu)}\,\psi^{(0)(+)}_{N_1}}\,\gamma^c\,\psi^{(0)(-)}_{N_2}\Big)\,,\end{aligned} \tag{4.29}$$

$$\begin{aligned}\frac{1}{2}\,K^{(0)\,\lambda\rho}_{\mu\nu}\,g^{(2)}_{(cl)\,\lambda\rho} = &-\sum_{|N|<N_0} R^{(0)(2)}_{\mu\nu}\,(h^*_N,\,h_N) + \\ &+\frac{i}{4}\sum_{|N|<N_F}\Big(\overline{\psi}^{(0)(-)}_{N}\,\gamma^c\,e^{(0)}_{c(\mu}\,D^{(0)}_{\nu)}\,\psi^{(0)(-)}_{N} - \\ &-e^{(0)}_{c(\mu}\overline{\mathcal{D}^{(0)}_{\nu)}\,\psi^{(0)(-)}_{N}}\,\gamma^c\,\psi^{(0)(-)}_{N}\Big)\,.\end{aligned} \tag{4.30}$$

It is evident from Eq. (4.2) that torsion appears in the same order $\sim l_P^2$. Here, however, we do not write out the corresponding corrections for the connection.

We shall now briefly summarize the results obtained.

According to the dynamic quantization method, the quantization of gravity starts with finding a solution of the classical microscopic field equations of motion (for example, the solution of Eqs. (4.15) in the example considered above). The classical solution is determined by (or determines) the topology of space-time. Then, using the classical approach, Eqs. (4.16) and (4.19), which determine the single-particle modes $\{\,\psi^{(\pm)}_N\,,\,h_{N\,\mu\nu}\}$, are solved. To solve Eq. (4.19) the gauge must be fixed, since the operator (4.17) is degenerate because of the gauge invariance of the theory. At the first step these modes are determined in the zeroth approximation according to the Planck scale, and their normalization is fixed using Eqs. (4.12) and (4.25). Given the set of modes $\{\,\psi^{(0)(\pm)}_N\,,\,h_{N\,\mu\nu}\}$, we can explicitly write out the right-hand sides of Eqs. (4.26)–(4.30) and then solve them for the two-particle modes $h_{N_1\,N_2\,\mu\nu}$, $h_{N_1\,|N_2\,\mu\nu}$, and so on, and find the correction $g^{(2)}_{(cl)\,\mu\nu}$ which is of second order in l_P to the classical component of the metric tensor. We call attention to the fact that the right-hand side of Eq. (4.30) arises because the operators must be normal-ordered. The solution of Eq. (4.30) can be interpreted as a single-loop contribution to the average of the metric tensor with respect to the ground state.

We note that if a nonsymmetric bilinear form were used on the right-hand sides of Eqs. (4.26)–(4.30), then the condition that the metric tensor be real would be violated. Consequently, the condition that the metric tensor is real determines the ordering of the operator fields in the equations of motion at least in second order with respect to the operator fields.

It is important that all Eqs. (4.15), (4.19), and so on which arise are generally covariant, since they are expansions of generally covariant equations. Thus, the dynamic quantization method leads to a regularized gauge-invariant theory of gravity, which contains an arbitrary number of physical degrees of freedom.

We shall now make a remark about the compatibility of Eqs. (4.26)–(4.30) and the analogous equations arising in higher orders. Let $h_{\mu\nu}$ be an arbitrary symmetric tensor field and $K^{(0)}$ the operator (4.17), acting on this tensor field. It is easily verified that, using Eqs. (4.15), we obtain the identity (compare with Eq. (4.21))

$$\nabla^{(0)}_\nu\,(K^{(0)}h)^\nu_\mu - \frac{1}{2}\nabla^{(0)}_\mu\,(K^{(0)}h)^\nu_\nu = 0\,. \tag{4.31}$$

Consequently, in order for Eqs. (4.26)–(4.30) to be compatible the right-hand sides of these equations must satisfy the same identity. It is easy to see that this is indeed the case in lowest order. Indeed, Eqs. (4.26)–(4.29) are identical to the analogous classical equations arising when nonuniform modes (higher order harmonics) and the subsequent expansion of the classical Einstein equation in powers of the nonlinearity or the Planck length are added to the uniform fields. Hence it follows that each term on the right-hand sides of the "loop" equations of the type (4.30) likewise satisfy the necessary identity, since these terms have the same form as the right-hand sides of the "nonloop" Eqs. (4.26)–(4.29).

In highest orders in creation and annihilation operators the compatibility of arising equations follows from the gauge invariance of the regularized Einstein equation. Indeed, the identity (4.31) is the consequence of gauge invariance (invariance relative to the general coordinate transformations) of the equation. To clarify the quation let's rewrite the action (4.1) (for simplicity with $m = 0$, $\Lambda = 0$) in the following form:

$$S = S_g + S_\psi\,, \tag{4.32}$$

$$S_g = -\frac{1}{4l_P^2}\int \mathrm{d}^4\,x\varepsilon_{abcd}\varepsilon^{\mu\nu\lambda\rho}e^a_\mu R^{bc}_{\nu\lambda}e^d_\rho\,, \tag{4.33}$$

$$S_\psi = \frac{1}{6}\int \mathrm{d}^4\, x\varepsilon_{abcd}\varepsilon^{\mu\nu\lambda\rho}\left[\frac{i}{2}\overline{\psi}e^b_\nu e^c_\lambda e^d_\rho \gamma^a D_\mu \psi + h.c.\right] \equiv$$
$$\equiv \int \mathrm{d}^4\, x\overline{\psi}\overleftrightarrow{\mathcal{D}}\,\psi\,. \qquad (4.34)$$

Here $\overleftrightarrow{\mathcal{D}}$ is Dirac hermithian operator, depending on other operator fields. The Heisenberg–Dirac equations are written in the form

$$\overrightarrow{\mathcal{D}}\,\psi = 0\,, \qquad \overline{\psi}\,\overleftarrow{\mathcal{D}} = 0\,. \qquad (4.35)$$

In Eqs. (4.35) the disposition of creation and annihilation operators is the same as in the action (4.32). Einstein equation is the condition of stationarity of the action (4.32) relative to variations of metric or tetrad. Evidently, the action (4.32) is invariant under the general coordinate transformation even if the fields are quantized. This follows from the facts that under the coordinate transformations all fundamental fields transform linearly and that the action (4.32) is a polynomial relative to the fundamental fields. Therefore, if the material fields are on mass shell (in our case this means that Eqs. (4.35) hold), the action (4.32) is stationary under infinitesimal gauge transformation of tetrad field only. This means that the quantum energy-momentum tensor on mass shell satisfies to some identity which in classical limit transforms to the well known identity $T^\mu_{\nu;\,\mu} = 0$. From this quantum identity it follows that if some quantum tetrad field satisfies Einstein equation, then the field transformed by infinitesimal gauge transformation also satisfies Einstein equation. From here the compatibility of quantum Einstein equation follows, as well as the compatibility of the chain of equations described above. However, this conclusion is true only if quantum Dirac equations (4.35) hold, and the operators in the action and energy-momentum tensor are placed so as in Eq. (4.32). In other words, the creation and annihilation operators in Eqs. (4.32) and (4.35) must be placed identically. This is the guarantee of self-consistency of the chain of equations arising from exact quantum Einstein and motion equations.

We also call attention to the fact that in the dynamic quantization method it is implicitly assumed that the quantum anomaly is absent in the algebra of the first class constraints operators. Consequently, the dynamic quantization method must be justified in each specific case by concrete calculations, which must be not only mathematically correct but also physically meaningful.

V. THE FUNDAMENTAL FIELDS AND THE SECONDARY QUANTIZED FIELDS

We see that the fundamental or cosmological fields are expanded in modes the number of which is finite in the case of compact space. Packing of the modes is essentially noncompact in momentum space. Assume that at present at low energies the minimal difference between the momenta of modes is of the order of $\Delta k_{min} \sim 1/\lambda_{\max}$. From the consideration at the end of Section 2 (see Eq. (2.56)) it follows that

$$\lambda_{\max} \sim \left(\frac{a_0}{a}\right)^{(\ln 3\lambda)/3\lambda} a \ll a\,. \qquad (5.1)$$

Further we denote by $a(t)$ the radius of universe and by t_0 the age of universe.

Further, one can assume that in considered theory the stochastization of phases of modes takes place on distances less than $\lambda_{\max}$. Under the phase stochastization we mean that any correlation between phases of wave packets spased by an enough distance can not take place. Such stochastization must occur if considered theory is the long-wavelength limit of discrete quantum theory of gravity discussed in Section 2. The point is that in long-wavelength limit the lattice action S transforms to the action which is expressed as follows:

$$S = S_{Einstein} + \Delta S\,.$$

Here $S_{Einstein}$ is standard Einstein action which does not retains any information about the structure of lattice, and ΔS depends only on higher derivatives of the fields and also it essentially depends on the structure of the lattice. Therefore equations of motion contain the items with higher derivatives of fields and casual coefficients depending on structure of irregular lattice (simplicial complex). These items play negligible part for low frequencies modes but their part increase with increasing of mode frequency. The items with higher derivatives of fields and casual coefficients lead to diffusional propagation of modes and so to stochastization of phase on large distances. But just due to this circumstance the high frequency wave packets can be localized in relatively small regions of space. This means that noncompact "packing" of modes in momentum space does not affects to the possibility of localization of high frequency wave packets.

One should pay attention to the fact that in presented theory with noncompact packing of modes the gravitational and gauge interaction forces does not become weaker. It is seen from quantum equations of motion which have the canonical form with usual interaction constants. Thus the interaction between any modes has the usual strength.

Let's consider, for example, the system of finite number of electrons, positrons and photons with wavelengths much less than λ_{max}. Assume that we are interested in the usual problem of particle physics: the scattering matrix problem. The dynamics of real relativistic particles is described by the usual Dirac and Maxwell equations. The dynamic process of particles localized in finite space volume $v \ll \lambda^3_{max}$ is studied. Since the matrix elements between localized states and nonlocalized states tends to zero as $a^{-3/2}(t_0)$, so only matrix elements between localized states are significant in the studied problem. This conclusion is true also with respect to virtual modes. From here it follows that for

description of processes proceeding in finite volume of space v, one must use the renormalized or secondary quantized quantum fields $(\psi_r, \ldots)$ which are normalized to the volume v. This means that the wave functions of the states $\{\psi_{r\,N}(x), \ldots\}$ which create and annihilate the localized particles are normalized to the volume v, these wave functions form the complete set of one particle wave functions with confined energies, and the corresponding creation and annihilation operators satisfy to standard relations (4.9). Note that the quantization conditions (4.9), i.e. nullification of ground state by annihilation operators, follow from the fact that the causal correlators $\langle 0\,|\,T\,\psi(x)\,\overline{\psi}(y)\,|0\,\rangle$ describe propagation omly positive-frequency waves. It seems that more general and correct definition of ground state $|0\rangle$ instead of definition (3.16) or (4.9) is that the amplitudes

$$\langle 0\,|\,T\,\psi(x)\,\overline{\psi}(y)\,|\,0\,\rangle\,, \qquad \langle 0\,|\,T\,A_i(x)\,A_j(y)\,|0\,\rangle\,, \ldots \tag{5.2}$$

describe the propagation of only positive-frequency waves if the times x^0 and y^0 are close to the time t_0. Again the definition of vacuum depends on the moment of time t_0. At present the state of Universe is close to the ground state. One can say that the renormalized fields $(\psi_r, \ldots)$ are the secondary quantized fields with the complete (at confined energies) and normalized on volume v set of one-particle states $\{\psi_{r\,N}(x), \ldots\}$. Thus the cosmological fields $(\psi, \ldots)$ from which quantum global Einstein equation is composed and the secondary quantized fields $(\psi_r, \ldots)$ are different though they describe the particles with the same quantum numbers. The causal correlators constructed from renormalized fields ψ_r (renormalized correlators) and thus describing local interactions also satisfy the conditions (5.2). Since local states normalized to volume v have compact "packing" in momentum space (at least for experimentally tested momenta), the renormalized correlators satisfy the standard equations:

$$(i\gamma^\mu\partial_\mu - m)\,\langle 0\,|\,T\,\psi(x)\,\overline{\psi}(y)\,|\,0\,\rangle = i\,\delta^4(x-y)\,, \ldots\,,$$

which are true at $|x^0 - y^0| \gg l_P,\ |\,\mathbf{x}-\mathbf{y}\,| \gg l_P$. And since the calculations of S-matrix elements are performed by using the standard Dirac and Maxwell equations with usual value of charge and others parameters, as a result the usual expressions for S-matrix elements are obtained.

Let's emphasize that at solving cosmological problems the retarded Green functions are used but at calculating S-matrix elements the causal or Feynman one are used.

Does the Casimir effect survives in proposed theory? The answer to this question is positive. Indeed, the attraction force between plates of condenser which is caused by Casimir effect is the derivative of sum of photon zero-point energies with respect to distance between plates. But only modes with wavelength commensurable with the distance between plates d really give the contribution in this derivative. And since $d \ll \lambda_{max}$ the distortion of Casimir effect does not occurs because this interaction is described by the secondary quantized fields.

We make the last remark about violation of Lorentz invariance in the theory. Since as a matter of fact the regularization is performed here by energy but not Lorentz invariant square of 4-momentum, so Lorentz invariance can be violated. However, until the processes with low energies (in comparison with the cutoff energy) are studied the violation of Lorentz invariance is negligible. The regularization by energy of calculations in QED is used, for example, in [18], and at the same time Lorentz invariance is not violated at low energies. Therefore the fact that all observed phenomena in nature are Lorentz covariant does not contradicts to the proposed theory since these phenomena has been observed at confined energies.

VI. THE POSSIBLE SOLUTION OF COSMOLOGICAL CONSTANT PROBLEM

It follows from Eqs. (4.16) that in lowest order the Dirac field

$$\psi^{(1)}(x) = \sum_{|N|<N_0} \left(a_N \psi_N^{(0)(+)}(x) + b_N^\dagger \psi_N^{(0)(-)}(x) \right) \tag{6.1}$$

satisfies the Dirac equation

$$(ie_a^{(0)\,\mu}\gamma^a D_\mu^{(0)} - m)\psi^{(1)}(x) = 0\,. \tag{6.2}$$

Here $D_\mu^{(0)}$ is the covariant derivative operator in zeroth order:

$$D_\mu^{(0)} = \partial/\partial x^\mu + (1/2)\omega_{ab\mu}^{(0)}\sigma^{ab} + ieA_\mu^{(0)}\,, \tag{6.3}$$

and A_μ is the gauge field.

It follows from Eq. (6.2) that the charge

$$Q = \int \mathrm{d}^3\,x\sqrt{-g^{(0)}}\,e_a^{(0)0}(\overline{\psi}^{(1)}\gamma^a\psi^{(1)}) \tag{6.4}$$

conserves (compare with (4.12)).

According to (4.6) the contribution of the Dirac field to the energy-momentum tensor in lowest order is equal to

$$T_{\psi\,\mu\nu}^{(2)} = \mathrm{Re}[i\overline{\psi}^{(1)}\gamma^a e_{a(\mu}^{(0)}D_{\nu)}^{(0)}\psi^{(1)}]\,. \tag{6.5}$$

Using Eqs. (4.9) it is easy to find vacuum expectation value of the quantity (6.5):

$$\langle T_{\psi\,\mu\nu}^{(2)}\rangle_0 = \mathrm{Re}\left[i\sum_{|N|<N_0} \overline{\psi}_N^{(0)(-)}\gamma^a e_{a(\mu}^{(0)}D_{\nu)}^{(0)}\psi_N^{(0)(-)} \right]. \tag{6.6}$$

Now let us take into account that the scenario described by the inflation theory is realized in Universe. It follows from here in conjunction with the used quantization method that in zeroth approximation the metric is expressed as

$$\mathrm{d}\,s^{(0)\,2} = \mathrm{d}\,t^2 - a^2(t)\;\mathrm{d}\,\Omega^2\,, \tag{6.7}$$

where $d\,\Omega^2$ is the metric on unite sphere S^3, and $a(t)$ is the scale factor of Universe at the running moment of time t. It follows from (6.7) that $e_a^{(0)0} = \delta_a^0$ and $\sqrt{-g^{(0)}}\,d^3\,x = d\,V^{(0)}(t)$, where $d\,V^{(0)}(t)$ is the volume element of 3-space in the running moment of time. From conservation of operator (6.4) the conservation of the set of integrals

$$\int d\,V^{(0)}(t)\,\psi_N^{(0)(\pm)\dagger}\psi_M^{(0)(\pm)} = \delta_{NM} \tag{6.8}$$

follows. The equality to unity of integrals (6.8) means that the wave functions $\psi_M^{(0)(\pm)}$ are normalized relative to the volume of all Universe, so that the charge operator has the form

$$Q = \sum_{|N|<N_0} (a^\dagger a_N + b_N b_N^\dagger)\,. \tag{6.9}$$

The idea how the vacuum expectation value of the matter energy-momentum tensor becomes enough small at present is demonstrated by the following estimation.

According to (6.8) we have:

$$\left|\overline{\psi}_N^{(0)(\pm)}\psi_N^{(0)(\pm)}\right| \sim \frac{1}{a^3(t)}\,. \tag{6.10}$$

Therefore the estimation for the value (6.6) is the following:

$$\langle T_{\psi\,\mu\nu}^{(2)}\rangle_0 \sim \frac{N_0 k_{max}}{a^3(t)}\,, \tag{6.11}$$

where k_{max} is the value of the order of maximal momentum of the modes $\{\psi_N^{(0)(\pm)}\}$. It is naturally to suppose that

$$k_{max} \sim l_P^{-1} \sim G^{-1/2} \sim 10^{33} cm^{-1}\,. \tag{6.12}$$

Since the numerator in the right hand side of relation (6.11) is finite and the denominator is proportional to the volume of Universe which swells up approximately 10^{100} times more according to inflation scenario, the quantity (6.11) can be found enough small at present.

On the other hand, it is seen from the estimation (6.11) that at early stages of Universe evolution the quantum fluctuations played decisive role because the scale of the Universe were small.

One should pay the attention to the fact that the dynamics of the system creates two opposite tendencies for mode frequencies changing.

According to the first tendency the frequencies ω of all one-particles modes change in time according to the low

$$\omega \sim \frac{1}{a(t)}\,. \tag{6.13}$$

The low (6.13) is valid in relativistic case. So, all frequencies decrees with expansion of Universe.

Now let us write out the first items of formal solution of Dirac equation (4.5) or (4.35) neglecting gravity degrees of freedom (i.e. in the case of flat space-time) but in the presence of gauge field:

$$\begin{aligned}\psi(x) = \psi^{(1)}(x) + e\int d^4\,y\,S_{ret}(x-y)\,A_\mu^{(1)}(y)\gamma^\mu\psi^{(1)}(y) + \\ +4\pi\,e^2\iint d^4\,y\,d^4\,z\,S_{ret}(x-y)\,D_{ret}(y-z)\times \\ \times\left(\overline{\psi}^{(1)}(z)\gamma_\mu\psi^{(1)}(z)\right)\gamma^\mu\psi^{(1)}(y) + \ldots\,,\end{aligned} \tag{6.14}$$

$$(i\gamma^\mu\partial_\mu - m)\,S_{ret}(x) = \delta^{(4)}(x)\,, \tag{6.15}$$

$$\partial_\mu\partial^\mu D_{ret}(x) = \delta^{(4)}(x)\,. \tag{6.16}$$

Here $S_{ret}(x)$ and $D_{ret}(x)$ are the retarded Green functions satisfying Eqs. (6.15) and (6.16). It is seen from the solution (6.14) that the exact field $\psi(x)$ have much more nonzero Fourier components than the field $\psi^{(1)}$. Hence, the exact solution of quantum Dirac equation has all Fourier components despite the field of first approximation $\psi^{(1)}$ has Fourier components only with finite momenta. From here we see the opposite dynamic tendency: the frequencies of modes effectively increases as a consequence of interaction. If the fact of strict conservation of the charge is taken into account [21], the conclusion about noncompact "packing" of modes in momentum space should be made. Indeed, let's calculate the mean value of charge operator relative to a state $|\,\rangle$. We have:

$$\int d^3\,x\langle\,|\,\psi^\dagger(x)\psi(x)\,|\,\rangle = \int \frac{d^3\,k}{(2\,\pi)^3}\langle\,|\,\psi_{|\,\mathbf{k}\,|}^\dagger\psi_{|\,\mathbf{k}\,|}\,|\,\rangle = \mathrm{const}\,,$$

$$\psi_{|\,\mathbf{k}\,|} = \int d^3\,x\,e^{-i\,\mathbf{k}\,\mathbf{x}}\psi(x)\,.$$

The last relation means also that the integral

$$\int d^3\,x\;\mathrm{tr}\langle\,|\,T\,\psi(x)\overline{\psi}(y)\,|\,\rangle\,\gamma^0\,,$$

$$y \longrightarrow x\,, \qquad y^0 > x^0$$

constructed with the help of correlator $\langle\,|\,T\,\psi(x)\overline{\psi}(y)\,|\,\rangle$ is conserved in time. But the mean value of energy-momentum tensor is constructed with the help of the same correlator. From here it is seen the effect of "loosening of mode packing" in momentum space. This effect is absent in the theory with dense packing of modes in all diapason of momenta since all states in momentum space are filled by the corresponding modes.

To solve the problem of "loosening of mode packing" in momentum space one must solve quantum kinetic equation for state density in momentum space. This problem is not solved in this work. But the fact of noncompact

"packing" of modes in momentum space plays an important role in our consideration. Thus, the noncompact "packing" of modes in momentum space is taken here as an assumption.

At a dense "packing" of modes in momentum space the neighbouring momenta differ by the quantity of the order of $\Delta k_{min} \sim 1/a(t)$. Therefore

$$\mathrm{d}\, N \sim \frac{a^3(t)\, \mathrm{d}^3\, k}{(2\,\pi)^3}\,. \tag{6.17}$$

At noncompact "packing" of modes in momentum space the neighbouring momenta differ by the greater quantity. Assume that at small momenta this difference at present is of the order of $\Delta k_{min} \sim 1/\lambda_{\max}$. Furthermore, we shall use Lorentz-invariant measure in momentum space $[\mathrm{d}^3\, k/|\, \mathbf{k}\,|]$. Thus we obtain instead of (6.17) the following estimation for the total number of physical degrees of freedom:

$$N_0 \sim \lambda_{max}^3 \int^{k_{max}} \frac{\mathrm{d}^3\, k}{(2\pi)^3 (\lambda_{max}|\, \mathbf{k}\,|)} \sim (\lambda_{max} k_{max})^2\,. \tag{6.18}$$

Now using (6.11) and (6.18) we find:

$$16\pi G \langle T_{\mu\nu}\rangle_0 \sim \frac{l_P^2\, \lambda_{max}^2\, k_{max}^3}{a^3(t)} \leq \Lambda\,, \tag{6.19}$$

and from here

$$a(t_0) \geq \frac{(l_P\, \lambda_{max})^{2/3}\, k_{max}}{\Lambda^{1/3}}\,. \tag{6.20}$$

If one assume that

$$\lambda_{max} \sim 10^{24} cm \sim 10^{-4} L\,, \tag{6.21}$$

where $L = 10^{28} cm$ (the dimension of observed part of Universe), then with the help of relations (1.2), (1.4), (6.21) and (6.20) we find the following estimation for the present dimension of Universe:

$$a(t_0) \geq 10^{17} L\,. \tag{6.22}$$

At obtaining the estimation (6.22) it was assumed that the fundamental field theory is not supersymmetric. If one assume that the fundamental theory is supersymmetric, but the spontaneous breaking of supersymmetry occurs on the momentum $\sim k_{SS}$, then the estimation of the dimension of Universe is changed. Indeed, in this case instead of (6.12) we have

$$k_{max} \sim k_{SS}\,, \tag{6.23}$$

since according to (1.9) and (1.10) the boson and fermion contributions to the vacuum expectation value of energy-momentum tensor with momenta greater than k_{SS} are mutually cancelled. Therefore instead of (6.20) we obtain:

$$a(t_0) \geq \frac{(l_P\, \lambda_{max})^{2/3}\, k_{SS}}{\Lambda^{1/3}} \sim 10^{29} cm \sim 10 L\,. \tag{6.24}$$

At obtaining the numerical estimation of right hand side Eq. (6.24) we used assumptions (6.21) and the popular assumption in particle physics that $k_{SS} \sim 10^3 GeV \sim 10^{17} cm^{-1}$.

The inclusion of quantum fluctuations of others fields into our estimations does not changes the result. This is clear already from dimensional considerations.

The inclusion of higher order corrections by perturbation theory also does not changes the obtained estimations. Indeed, all known fundamental interactions except for gravitational are renormalizable and thus can be considered by perturbation theory without changing fundamental properties of the vacuum. But the gravitational quantum corrections are obtained by expanding in Planck scale l_P. Again from dimensional considerations it is clear that such corrections at passing to the following order in our theory have the comparative value

$$\sim \left(\frac{l_P}{a(t_0)}\right)^2 N_0 \sim \\ \sim \left(\frac{l_P}{a(t_0)}\right)^2 (\lambda_{max} k_{max})^2 \leq \left(\frac{\lambda_{max}}{a(t_0)}\right)^2 \ll 1\,. \tag{6.25}$$

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APPENDIX A

Let us consider the discrete Laplace operator on a one dimensional cycle with 3 vertexes (see fig. 1). The numbers a, b, c are the distances between the vertexes 1 and 2, 2 and 3, 3 and 1, correspondingly. In the vertexes 1, 2 and 3 the real numbers φ_1, φ_2 and φ_3 are defined. Write out the discrete equation for Laplace operator eigenfunctions:

$$\begin{aligned} -(\Delta\varphi)_1 &= -\frac{2}{ac}\left(\frac{a\,\varphi_3 + c\,\varphi_2}{a+c} - \varphi_1\right) = \epsilon\,\varphi_1\,, \\ -(\Delta\varphi)_2 &= -\frac{2}{ab}\left(\frac{b\,\varphi_1 + a\,\varphi_3}{a+b} - \varphi_2\right) = \epsilon\,\varphi_2\,, \\ -(\Delta\varphi)_3 &= -\frac{2}{bc}\left(\frac{c\,\varphi_2 + b\,\varphi_1}{b+c} - \varphi_3\right) = \epsilon\,\varphi_3\,. \end{aligned} \tag{A1}$$

For slowly varying from vertex to vertex variables φ_i the system of equations (A1) transforms to the continual equation $-\Delta\,\varphi = \epsilon\,\varphi$. The three eigenvalues of Eq. (A1)

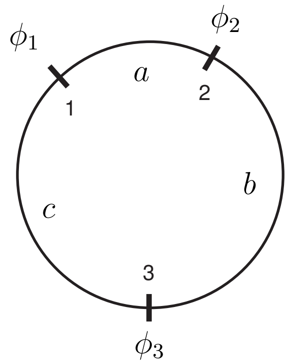

FIG. 1:

are as follows:

$$\epsilon_1 = 0\,,$$
$$\epsilon_{2,3} = \frac{a+b+c}{abc}\left[1 \pm \sqrt{1 - \frac{8\,abc}{(a+b)(a+c)(b+c)}}\right]. \tag{A2}$$

If $a \to 0$ and $(a+b+c) = \text{const}$, then

$$\epsilon_2 \sim \frac{2(b+c)}{abc} \longrightarrow \infty\,, \qquad \epsilon_3 = \frac{4}{bc}\,. \tag{A3}$$

Consider the same problem for the discrete Laplace operator on a one dimensional cycle with 4 vertexes separated in order by distances a, b, c and d. Then the eigenvalues of the operator satisfy the following equation:

$$\begin{aligned}
&\epsilon^4 - 2\epsilon^3\left(\frac{1}{cd} + \frac{1}{bc} + \frac{1}{ab} + \frac{1}{ad}\right) + \\
&+4\epsilon^2\left[\frac{1}{bc^2d} + \frac{1}{ab^2c} + \frac{1}{acd^2} + \frac{1}{a^2bd} + \frac{2}{abcd} - \right.\\
&-\frac{1}{c^2(b+c)(c+d)} - \frac{1}{b^2(a+b)(b+c)} - \\
&\left.-\frac{1}{a^2(a+b)(a+d)} - \frac{1}{d^2(a+d)(c+d)}\right] - \\
&-8\epsilon\left[\frac{1}{ab^2c^2d} + \frac{1}{abc^2d^2} + \frac{1}{a^2bcd^2} + \frac{1}{a^2b^2cd} - \right.\\
&-\frac{a+c}{ab^2cd(a+b)(b+c)} - \frac{b+d}{a^2bcd(a+d)(a+b)} - \\
&\left.-\frac{b+d}{abc^2d(b+c)(c+d)} - \frac{a+c}{abcd^2(a+d)(c+d)}\right] = 0\,.
\end{aligned} \tag{A4}$$

Though the exact solution we do not obtained the approximate solutions of this equation in two interesting here special cases are written out:

$$\begin{gathered}
b = d = l\,, \qquad a \to 0\,, \qquad c \to 0\ : \\
\epsilon_1 = 0\,, \quad \epsilon_2 \approx \frac{4}{l^2}\,, \\
\epsilon_3 \approx \frac{4}{la} \to \infty\,, \quad \epsilon_4 \approx \frac{4}{lc} \to \infty\,.
\end{gathered} \tag{A5}$$

$$\begin{gathered}
c = d = l\,, \qquad a \to 0\,, \qquad b \to 0\ : \\
\epsilon_1 = 0\,, \quad \epsilon_{2,3} \approx \frac{2}{l(a+b)} \to \infty\,, \\
\epsilon_4 \approx \frac{2}{ab} - \frac{4}{l(a+b)} \to \infty\,.
\end{gathered} \tag{A6}$$

APPENDIX B

For clearness it is useful to see the phenomenon of imposing the second class constraints without changing of quantum field equations on an example of free Klein–Gordon theory. The Klein–Gordon fields are expanded as follows:

$$\begin{aligned}
\phi(x) &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}\phi_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}}\phi^*_{\mathbf{k}}(x)\right)\,, \\
\pi(x) &= -i\sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left(a_{\mathbf{k}}\phi_{\mathbf{k}}(x) - a^\dagger_{\mathbf{k}}\phi^*_{\mathbf{k}}(x)\right)\,, \\
\omega_{\mathbf{k}} &= \sqrt{\mathbf{k}^2 + m^2}\,, \qquad [a_{\mathbf{k}},\, a^\dagger_{\mathbf{p}}] = \delta_{\mathbf{k}\,\mathbf{p}}\,.
\end{aligned} \tag{B1}$$

Here $\{\phi_{\mathbf{k}}(x)\}$ is the complete set of orthonormal functions, so that

$$\begin{gathered}
\sum_{\mathbf{k}} \phi_{\mathbf{k}}(x)\,\phi^*_{\mathbf{k}}(y)\bigg|_{x^0=y^0} = \delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \\
\Delta\,\phi_{\mathbf{k}}(x) = -\,\mathbf{k}^2\ \phi_{\mathbf{k}}(x)\,.
\end{gathered} \tag{B2}$$

The Hamiltonian

$$\begin{aligned}
\mathcal{H} &= \int \mathrm{d}^3\,x\left(\frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi\nabla\phi + \frac{m^2}{2}\phi^2\right) = \\
&= \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(a_{\mathbf{k}}a^\dagger_{\mathbf{k}} + a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\right)\,.
\end{aligned} \tag{B3}$$

Equations of motions are obtained with the help of Eqs. (B1)–(B3):

$$\begin{gathered}
\dot\phi(x) = -i[\phi(x),\, \mathcal{H}] = \\
= -i\sum_{\mathbf{k}}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left(a_{\mathbf{k}}\phi_{\mathbf{k}}(x) - a^\dagger_{\mathbf{k}}\phi^*_{\mathbf{k}}(x)\right) = \pi(x)\,, \\
\ddot\phi(x) = \dot\pi(x) = -i[\pi(x),\, \mathcal{H}] = \\
= -\sum_{\mathbf{k}} \frac{\omega^2_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}\phi_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}}\phi^*_{\mathbf{k}}(x)\right) = (\Delta - m^2)\,\phi(x)\,.
\end{gathered} \tag{B4}$$

Now let us impose any number of pairs of second class constraints

$$a_{\mathbf{k}_i} = 0\,, \qquad a^\dagger_{\mathbf{k}_i} = 0\,, \qquad i = 1, 2\,\ldots\,. \tag{B5}$$

Then the sums $\sum_{\mathbf{k}}$ in (B1), (B3) and (B4) transform to the reduced sums $\sum_{\mathbf{k}\neq\mathbf{k}_i}$. Nevertheless equation of motion (B4) retains its canonical form

$$\left(\partial^2/(\partial x^0)^2 - \Delta + m^2\right)\,\phi(x) = 0\,. \tag{B6}$$

The dynamical reason for this conclusion in considered example is that the commutators of the constraints (B5) with Hamiltonian are proportional to the constraints, i.e. they are equal to zero in a weak sense:

$$[a_{\mathbf{k}_i}, \mathcal{H}] = \omega_{\mathbf{k}_i} a_{\mathbf{k}_i}\,, \qquad [a^{\dagger}_{\mathbf{k}_i}, \mathcal{H}] = -\omega_{\mathbf{k}_i} a^{\dagger}_{\mathbf{k}_i}\,. \tag{B7}$$

Therefore, as it was shown in Section 3, equations of motion in reduced theory must retain their canonical form.

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[19] Here, by an almost smooth surface, we mean a piecewise smooth surface consisting of flat four-dimensional simplices, such that the angles between adjacent 4-simplices tend to zero and the sizes of these simplices are commensurable.

[20] In Appendix B we give the simple example in which the imposition of second class constraints of type (3.13) does not change equations of motion.

[21] It means strict conservation of quantum charge operator which in lowest order transforms into expression (6.4)